

The background features a large, faint, circular seal of the University of Gothenburg. The seal contains a central figure holding a sword and a book, surrounded by Latin text: "SIGILLUM UNIVERSITATIS GOTHORVM CAROLINÆ" and the year "1666".

Nonlinear Control and Servo Systems

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Today's Goal

To be able to

- ▶ *prove local and global stability of an equilibrium point through Lyapunov's method*



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- ▶ *prove local and global stability of an equilibrium point through Lyapunov's method*
- ▶ *show stability of a set (for example, a limit cycle) through invariant set theorems*

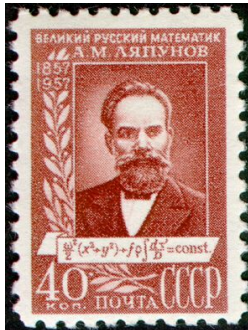


Material

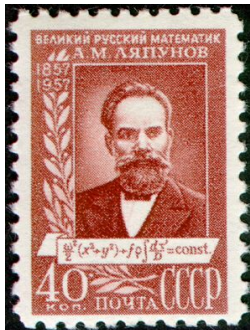
- ▶ Slotine and Li: Chapter 3
- ▶ Lecture notes



Alexandr Mihailovich Lyapunov (1857–1918)



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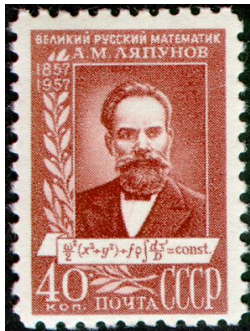


Master's thesis

“On the stability of ellipsoidal forms of equilibrium of rotating fluids,” St. Petersburg University, 1884.



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Doctoral thesis

“The general problem of the stability of motion,” 1892.



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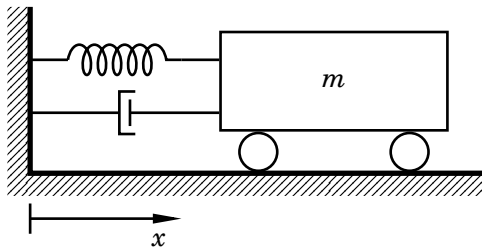
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Main question

How to find a Lyapunov function?



A Motivating Example



$$m\ddot{x} = - \underbrace{b\dot{x}|\dot{x}|}_{\text{damping}} - \underbrace{k_0x - k_1x^3}_{\text{spring}}, \quad b, k_0, k_1 > 0$$

The energy can be shown to be

$$V(x, \dot{x}) = m\dot{x}^2/2 + k_0x^2/2 + k_1x^4/4 > 0, \quad V(0, 0) = 0$$
$$\frac{d}{dt}V(x, \dot{x}) = m\dot{x}\ddot{x} + k_0x\dot{x} + k_1x^3\dot{x} = -b|\dot{x}|^3 < 0,$$



Stability Definitions

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locally stable, if for every $R > 0$ there exists $r > 0$, such that

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globally asymptotically stable, if asymptotically stable for all $x(0) \in \mathbf{R}^n$.



Lyapunov Theorem for Local Stability

Theorem

Let $\dot{x} = f(x)$, $f(0) = 0$, and $0 \in \Omega \subset \mathbf{R}^n$.

Assume that $V : \Omega \rightarrow \mathbf{R}$ is a C^1 function. If

- ▶ $V(0) = 0$
- ▶ $V(x) > 0$, for all $x \in \Omega$, $x \neq 0$
- ▶ $\frac{d}{dt}V(x) \leq 0$ along all trajectories in Ω

then $x = 0$ is locally stable. Furthermore, if also

- ▶ $\frac{d}{dt}V(x) < 0$ for all $x \in \Omega$, $x \neq 0$

then $x = 0$ is locally asymptotically stable.

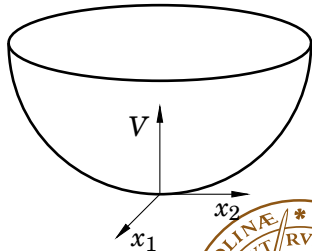
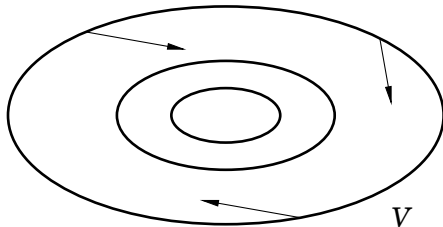
Proof: see p. 62.



Lyapunov Functions (\approx Energy Functions)

A Lyapunov function fulfills $V(x_0) = 0$, $V(x) > 0$ for $x \in \Omega$, $x \neq x_0$, and

$$\dot{V}(x) = \frac{d}{dt}V(x) = \frac{dV}{dx}\dot{x} = \frac{dV}{dx}f(x) \leq 0$$



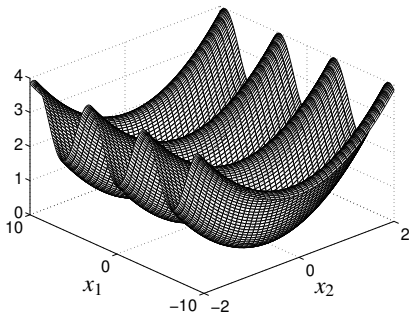
2 min exercise—Pendulum

Show that the origin is locally stable for a mathematical pendulum.

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{g}{\ell} \sin x_1$$

Use as a Lyapunov function candidate

$$V(x) = (1 - \cos x_1)gl + \ell^2 x_2^2/2$$



Lyapunov Theorem for Global Stability

Theorem Let $\dot{x} = f(x)$ and $f(0) = 0$. Assume that $V : \mathbf{R}^n \rightarrow \mathbf{R}$ is a C^1 function. If

- ▶ $V(0) = 0$
- ▶ $V(x) > 0$, for all $x \neq 0$
- ▶ $\dot{V}(x) < 0$ for all $x \neq 0$
- ▶ $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$

then $x = 0$ is globally asymptotically stable.

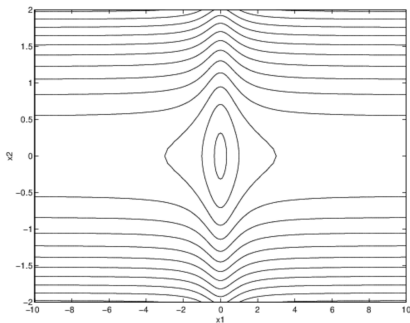


Radial Unboundedness is Necessary

If the condition $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$ is not fulfilled, then global stability cannot be guaranteed.

Example Assume $V(x) = x_1^2/(1+x_1^2) + x_2^2$ is a Lyapunov function for a system. Can have $\|x\| \rightarrow \infty$ even if $\dot{V}(x) < 0$.

Contour plot $V(x) = C$:



Positive Definite Matrices

A matrix M is **positive definite** if $x^T M x > 0$ for all $x \neq 0$.

It is **positive semidefinite** if $x^T M x \geq 0$ for all x .

A symmetric matrix $M = M^T$ is positive definite if and only if its eigenvalues $\lambda_i > 0$. (*semidefinite* $\Leftrightarrow \lambda_i \geq 0$)

Note that if $M = M^T$ is positive definite, then the Lyapunov function candidate $V(x) = x^T M x$ fulfills $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$.



More matrix results

A symmetric matrix $M = M^T$ satisfies the inequalities

$$\lambda_{\min}(M)\|x\|^2 \leq x^T M x \leq \lambda_{\max}(M)\|x\|^2$$

(To show it, use the factorization $M = U\Lambda U^*$, where U is a unitary matrix, $\|Ux\| = \|x\|$, U^* is complex conjugate transpose, and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$.)

For any matrix M one also has

$$\|Mx\| \leq \lambda_{\max}^{1/2}(M^T M)\|x\|$$



Lyapunov Function for Linear System

Theorem The eigenvalues λ_i of A satisfy $\operatorname{Re} \lambda_i < 0$ if and only if: for every positive definite $Q = Q^T$ there exists a positive definite $P = P^T$ such that

$$PA + A^T P = -Q$$

Proof of $\exists Q, P \Rightarrow \operatorname{Re} \lambda_i(A) < 0$: Consider $\dot{x} = Ax$ and the Lyapunov function candidate $V(x) = x^T P x$.

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P)x = -x^T Q x < 0, \quad \forall x \neq 0$$

$$\Rightarrow \dot{x} = Ax \quad \text{asymptotically stable} \quad \iff \quad \operatorname{Re} \lambda_i < 0$$

Proof of $\operatorname{Re} \lambda_i(A) < 0 \Rightarrow \exists Q, P$: Choose $P = \int_0^\infty e^{A^T t} Q e^{At} dt$



Lyapunov's Linearization Method

Recall from Lecture 2:

Theorem Consider

$$\dot{x} = f(x)$$

Assume that $x = 0$ is an equilibrium point and that

$$\dot{x} = Ax + g(x)$$

is a linearization.

- (1) If $\text{Re } \lambda_i(A) < 0$ for all i , then $x = 0$ is locally asymptotically stable.
- (2) If there exists i such that $\lambda_i(A) > 0$, then $x = 0$ is unstable.

