Structure Preserving H_∞-optimal Control

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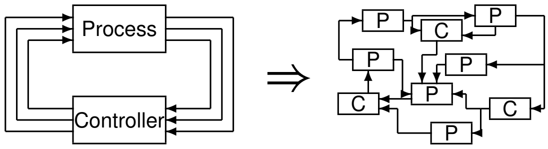
Towards a Scalable Control Theory



Today: Exploit (network) structure for scalability

Background

H_∞ **Control:**

Riccati solution: Doyle, Glover, Khargonekar and F (1989)
LMI solution: Packard (1994) and Apakarian/Gahinet (1995)

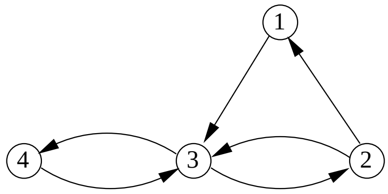
Optimization of structured controllers:

Spatially invariant systems: Bamieh, Paganini, Dahleh (2002)
Distributed controllers: D'Andrea and Dullerud (2003)
Quadratic invariance: Rotkowitz and Lall (2002)
Low rank coordination: Madjidian and Mirkin (2014)

Dsitributed control of positive systems:

Synthesis by linear programming: Rami and Tadeo (2007)
H_∞ control via LMIs: Tanaka and Langbort (2011)
Robust control: Briat (2013)
Scalability and positively dominated systems: Rantzer (2015)

Example: Network Flow Dynamics



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}-1-\ell_{31} & \ell_{12} & 0 & 0\\ 0 & -\ell_{12}-\ell_{32} & \ell_{23} & 0\\ \ell_{31} & \ell_{32} & -\ell_{23}-\ell_{43} & \ell_{34}\\ 0 & 0 & \ell_{43} & -4-\ell_{34}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}w_1\\ w_2\\ w_3\\ w_4\end{bmatrix}$$

How do we select ℓ_{ij} to optimize the dynamics?

Network Flow Dynamics in Practice

- Irrigation systems
- Power systems
- Traffic flow dynamics
- Communication/computation networks
- Production planning and logistics

Outline

- **H_∞ Optimal Structured Static Controllers**
- H_∞ Optimal Structured PI Controllers
- Graph Structure and Positivity

H_∞ Optimal Static Control on Networks

Problem:
Given a graph $(\mathcal{V}, \mathcal{E})$ and

$$\dot{x}_i = a_i x_i + \sum_{(i,j)\in\mathcal{E}} (u_{ij} - u_{ji}) + w_i \qquad i \in \mathcal{V}$$

find control law $u = Kx$ that minimizes the H_∞ norm of the map from w to (x, u).

Solution:
An optimal control law when $a_i < 0$ is given by

$$u_{ij} = x_i/a_i - x_j/a_j \qquad (i,j) \in \mathcal{E}.$$

[Lidström/Rantzer, ACC2016]

Structure Preserving Static Feedback

Problem
Consider the system $\dot{x} = Ax + Bu + w$ with A symmetric and Hurwitz. Find a state feedback controller $u = Kx$ that minimizes the H_∞ norm of the map from w to (x, u) in the closed loop system $\dot{x} = (A + BK)x + w$.

Theorem
A solution is given by $u = K_* x$ where $K_* = B^T A^{-1}$. The minimal value of the norm is $\sqrt{\|(A^2 + BB^T)^{-1}\|}$.

Remarks
No Riccati equation needs to be solved.
If A is diagonal and B is sparse, than also K_* is sparse.

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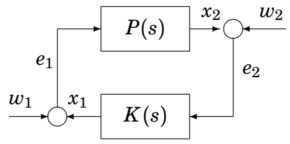
Frequency Weighted Specifications



$$e_1 = Ke_2 + w_1$$
$$e_2 = Pe_1 + w_2$$

Disturbance rejection:
The transfer functions $(I + PK)^{-1}$ and $(I + PK)^{-1}P$ should be small for low frequencies. ("Integral action")

Measurement errors:
The transfer functions $K(I + PK)^{-1}$ and $PK(I + PK)^{-1}$ should be small for high frequencies.

Structure Preserving Dynamic Feedback

Theorem
Let $P(s) = (sI - A)^{-1}B$, where $A \in \mathbb{R}^{n\times n}$ is symmetric negative definite. Define $F_K(s) = [I + K(s)P(s)]^{-1}K(s)$ and suppose that $\tau \geq \sqrt{\|B^TA^{-4}B\|}$.

Then the problem to minimize $\|F_K\|_\infty$ over stabilizing $K(s)$ subject to $\|\frac{1}{s}P(I + KP)^{-1}\|_\infty \leq \tau$ is solved by

$$\widehat{K}(s) = k\left(B^TA^{-2} - \frac{1}{s}B^TA^{-1}\right).$$

where $k = \tau^{-1}\|(A^{-1}B)^\dagger\|$.

The minimal value of $\|F_K\|_\infty$ is $\|(A^{-1}B)^\dagger\|$.

Proof Sketch

Recall the definition $F_K = (I + KP)^{-1}K$.

The constraint $\|\frac{1}{s}P(I + KP)^{-1}\|_\infty \leq \tau$ implies that $F_K(0)$ must satisfy $P(0)F_K(0)P(0) = P(0)$.

Minimizing $\|F\|$ subject to $P(0)FP(0) = P(0)$ gives the minimal value $\|(A^{-1}B)^\dagger\|$ attained by $F = -(A^{-1}B)^\dagger$. The given controller attains this value and

$$\left\|\tfrac{1}{i\omega}P(I + \widehat{K}P)^{-1}\right\| = \left\|(i\omega I - A)^{-1}B\left(i\omega I + kB^TA^{-2}B\right)^{-1}\right\|$$
$$\|F_{\widehat{K}}(i\omega)\| = \left\|k\left(i\omega + kB^TA^{-2}B\right)^{-1}B^TA^{-2}(i\omega I - A)\right\|$$

are both maximal a $\omega = 0$, so the optimal value is attained.

Optimal Network Control with Edge Integrators

Given a graph $(\mathcal{V}, \mathcal{E})$, let $P(s)$ be the transfer matrix from u to x given by

$$\dot{x}_i = a_ix_i + \sum_{(i,j)\in\mathcal{E}}(u_{ij} - u_{ji}) \qquad i \in \mathcal{V}$$

with $a_i < 0$. Then $\widehat{K}(s)$ is the dynamic map from x to u given by

$$\begin{cases} \dot{z}_{ij} = k(x_i/a_i - x_j/a_j) \\ u_{ij} = z_{ij} - x_i/a_i^2 + x_j/a_j^2 \end{cases}$$

A separate PI controller for each edge of the graph.
Works if the graph is a tree! ... What goes wrong otherwise?

Optimal Network Control with Node Integrators

Given a graph $(\mathcal{V}, \mathcal{E})$, let the plant be given by

$$\dot{x}_i = a_ix_i + b_iu_i + \sum_{(i,j)\in\mathcal{E}}(u_{ij} - u_{ji}) \qquad i \in \mathcal{V}$$

with $a_i < 0$. Then $\widehat{K}(s)$ is the map from x to u given by

$$\begin{cases} \dot{z}_i = x_i \\ u_{ij} = z_i/a_i - x_i/a_i^2 - z_j/a_j + x_j/a_j^2 \\ u_i = b_i(z_i/a_i - x_i/a_i^2) \end{cases}$$

There is a problem if all b_i are zero. Why?

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Limitations due to Graph Structure

Let $A = -I$ while B is an oriented incidence matrix, e.g.

$$B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

If the graph is a tree, B has full column rank and

$$\|F_{\widehat{K}}\|_\infty = \frac{1}{\sqrt{\|(B^TB)^{-1}\|}} = \frac{1}{\sqrt{\lambda_2}},$$

where λ_2 is the algebraic connectivity of the graph. In two extreme cases, a star graph and a path, λ_2 is

$$1 \quad \text{and} \quad \left(2\sin\left(\frac{\pi}{n}\right)\right)^2$$

respectively.

Closed Loop Positive Systems

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t) \qquad x(0) = 0$$

with A diagonal and negative definite, while $-BB^T$ is Metzler.

The proportional controller $u(t) = B^T A^{-1} x(t)$ makes the map from w to x positive.

The PI controller

$$\begin{cases} \dot{z}(t) = x(t) \\ u(t) = -k\left[B^T A^{-2} x(t) + B^T A^{-1} z(t)\right] \end{cases} \qquad z(0) = 0$$

makes the map from w to z positive.

For Networked Control — Use H_∞ Optimization!

- Structure preserving optimal PI controllers
- No need for global information
- More research needed
  - Output feedback
  - More general dynamics
  - Nonlinear extensions
- Many driving applications