Cheap Control of Delayed Transportation Systems

Martin Heyden

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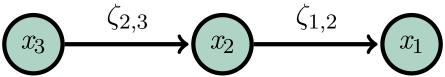
Lund University

Introduction - Dynamics

- We consider a simple transportation problem with delays on a Graph.
- For a node k with children set c_k and parent set p_k the dynamics are given by

$$\begin{aligned}\hat{x}_k[t+1] =& \hat{x}_k[t] + \sum_{i \in p_k} \hat{\zeta}_{k,i}[t] - \sum_{j \in c_k} \hat{u}_{j,k}[t] + \hat{w}_k \\ \hat{\zeta}_{k,i}[t+1] =& \hat{u}_{k,i}[t].\end{aligned}$$



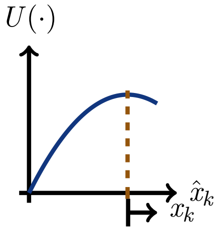
- $\hat{x}_k \geq 0$ inventory level in node k
- $\hat{u}_{j,k} \geq 0$ internal transportation, goods to be sent from node k to node j.
- $\hat{\zeta}_{k,i} \geq 0$ good in transit from node i to node k.
- $\hat{w}_K \in \mathcal{N}(\bar{w}, \sigma_k)$ external flow.

Introduction - Cost

- Each node have a utility function describing how much it values different inventory level. $U_k(\hat{x}_k) = q_k \hat{x}_k(a_k - \hat{x}_k)$.
- Goal is to maximize total utility, $\max_{\hat{x}} \sum_{t=0}^{\infty} \alpha^t \sum_i U_i(\hat{x}_i[t])$
  - No penalty on input (motivated by that there is a transportation independent of choice of u.)
  - Goods in transit have no utility.
  - Discounted optimization - Profits now are better than later. Can be replaced by a decay of goods.

Let $x_k = \hat{x}_k - a_k$.

$$U(x) = \min \sum_{t=0}^{\infty} \alpha^t \sum_k q_k x_k[t]^2$$



Problem Transformation

Work with nominal flow

- $\hat{u} = \bar{u} + u$, $\hat{\zeta} = \bar{\zeta} + \zeta$ and $\hat{w} = \bar{w} + w$.where $\bar{u}$ and $\bar{\zeta}$ are the nominal internal flow and $\bar{w}$ are nominal external flows.
- $u \geq -\bar{u}$ can take negative values

Then dynamics are given by

$$\begin{aligned} x_k[t+1] =& x_k[t] + \sum_{i \in p_k} \zeta_{k,i}[t] - \sum_{j \in c_k} u_{j,k}[t] + w_k \\ \zeta_{k,i}[t+1] =& u_{k,i}[t]. \end{aligned}$$

With cost

$$\min \sum_{t=0}^{\infty} \alpha^t \sum_k q_k x_k[t]^2$$

Standard (discounted) LQ problem!!

Line Results

Recursively define γ_k as

$$\gamma_{k+1} = \alpha \frac{q_{k+1}\gamma_k}{q_{k+1} + \gamma_k}, \quad \gamma_1 = \alpha q_1$$

And let

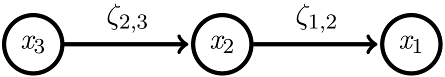
$$\sigma_k = \frac{\gamma_{k-1}}{q_k + \gamma_{k-1}}.$$

Then the optimal input is given by

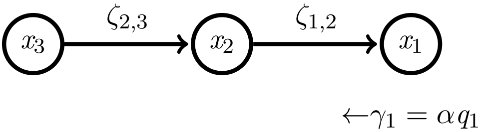
$$u_{k,k+1} = (1 - \sigma_k)(x_{k+1} + \zeta_{k,k+1}) - \sigma_k \underbrace{\sum_{i=1}^{k} (x_i + \zeta_{i,i+1})}_{\text{Goods downstream}}$$

- Efficient recursive calculation of feedback law.
- Sparse!
- Efficient local communication possible.
- Can be updated easily as the graph shrinks or growths in one direction.

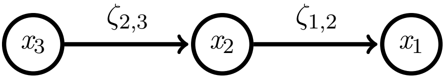
Example - Calculate gains



Example - Calculate gains

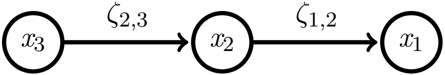


Example - Calculate gains



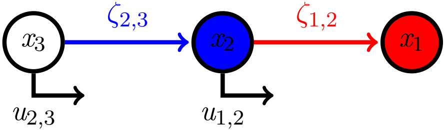
$$\leftarrow \gamma_2 = \alpha \frac{q_2 \gamma_1}{q_2 + \gamma_1}, \quad \leftarrow \gamma_1 = \alpha q_1$$

Example - Calculate gains



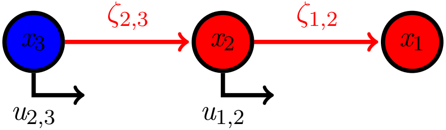
$$\gamma_3 = \alpha \frac{q_3 \gamma_2}{q_3 + \gamma_2}, \leftarrow \gamma_2 = \alpha \frac{q_2 \gamma_1}{q_2 + \gamma_1}, \ \leftarrow \gamma_1 = \alpha q_1$$

Example - Calculate u



$$u_{1,2} = (1 - \sigma_2)(x_2 + \zeta_{2,3}) - \sigma_2(x_1 + \zeta_{1,2})$$

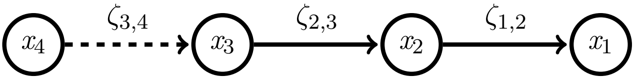
Example - Calculate u



$$u_{1,2} = (1 - \sigma_2)(x_2 + \zeta_{2,3}) - \sigma_2(x_1 + \zeta_{1,2})$$

$$u_{2,3} = (1 - \sigma_3)x_3 - \sigma_3(x_2 + \zeta_{2,3} + x_1 + \zeta_{1,2})$$

Example - Extending the Graph



$\gamma_4 = \alpha \frac{q_4 \gamma_3}{q_4 + \gamma_3}$, $\leftarrow \gamma_3$ γ_2 γ_1

Solution Sketch

- Let $v_{i,j}$ be node i choice of what to receive from node j. Then we can write the amount of goods in each node independently.
- However $u_{i,j} = v_{i,j}$. Constrained optimization!

We can show that for the constrained finite horizon problem

$$\mathcal{L}(u, v, \lambda) = \sum_{k \in \mathcal{N}} V_k(u_{k-1,k}, v_{k,k+1}, \lambda_{k-1,k}, \lambda_{k,k+1})$$

$$\frac{\partial}{\partial u_{j,k}} V_k = 0$$
$$\frac{\partial}{\partial v_{k,i}} V_k = 0$$
$$u = v$$

Solution sketch

$$\begin{cases} \frac{\partial V_1}{\partial v_{1,2}} = 0 \Leftrightarrow 2q_1\alpha(v_{1,2}[0] + x_1[0] + \zeta_{1,2}[0]) & = \lambda_{1,2}[0] \\ \frac{\partial V2}{\partial u_{1,2}} = 0 \Leftrightarrow 2q_2(u_{1,2}[0] - x_2[0] - \zeta_{2,3}[0]) & = -\lambda_{1,2}[0] \end{cases}$$

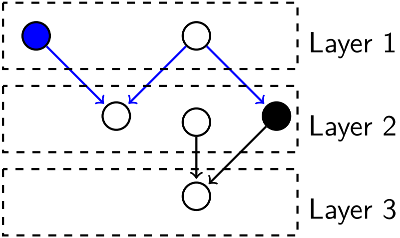
- This can easily be solved and gives a state feedback law for $u_{1,2}[0]$
- The solution is the same for any time horizon $N > 1$
- So we find a state feedback for $u_{1,2}[t]$ for $t < N - 1$

For the next link we have that:

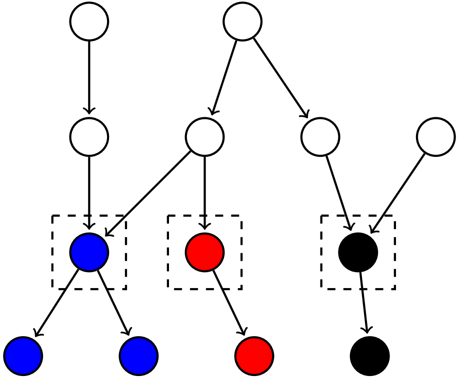
$$\begin{cases} 2q_2(v_{2,3}[0] - u_{1,2}[0] - u_{1,2}[1] + x_2[0] + \zeta_{2,3}[0]) & = \lambda_{2,3}[0] \\ 2q_3(u_{2,3}[0] - x_3[0] - \zeta_{3,4}[0]) & = -\lambda_{2,3}[0] \end{cases}$$

- Independent of upstream!
- Important observation: $u_{1,2}[1] = (1 - \sigma_1)u_{2,3}[0]$

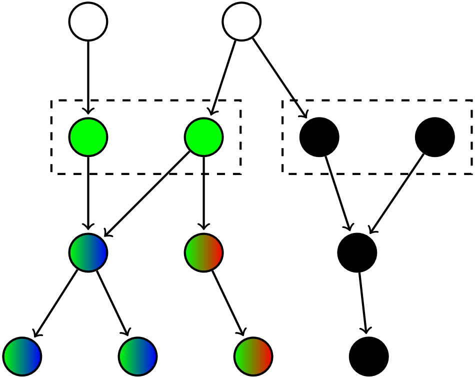
Graph Partitioning - Layers



Graph Partitioning - Blocks



Graph Partitioning - Blocks



Results - Polytree

For a block a with children block set $\mathcal{C}$

$$\gamma_a = \frac{1}{\sum_{i \in a} \frac{1}{q_i} + \sum_{j \in \mathcal{C}} \frac{1}{\gamma_j}}$$

For a node k in block a

$$\sigma_a = \frac{\gamma_a}{q_j}$$

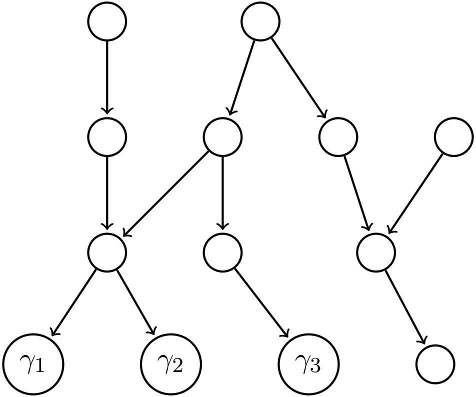
Then output of the node is given by

$$u_k = (1 - \sigma_a) X_k - \sigma_a (\mathcal{X}_a - X_k)$$

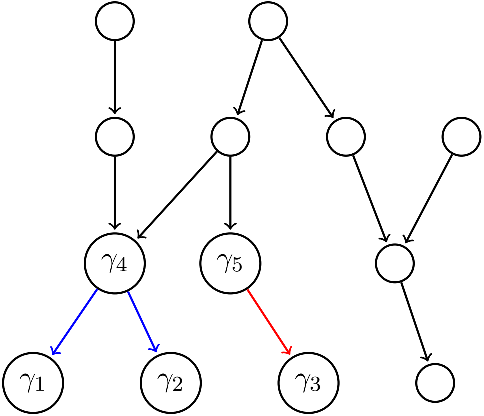
Where $X_k = x_k + \sum_{i \in p} \zeta_{k,i}$ and $\mathcal{X}_a$ is the total mass in block a and downstream.

There is a similar expression for input into the nodes.

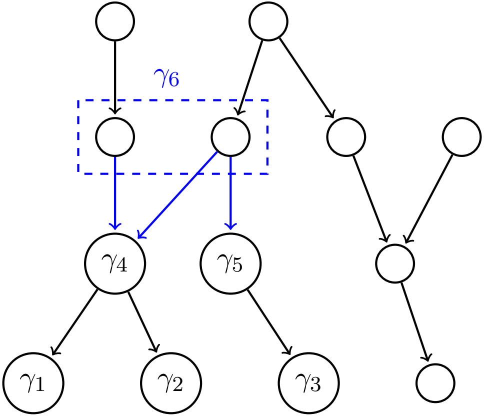
Gamma iteration



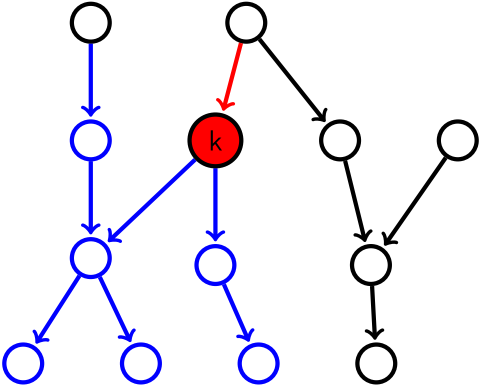
Gamma iteration



Gamma iteration

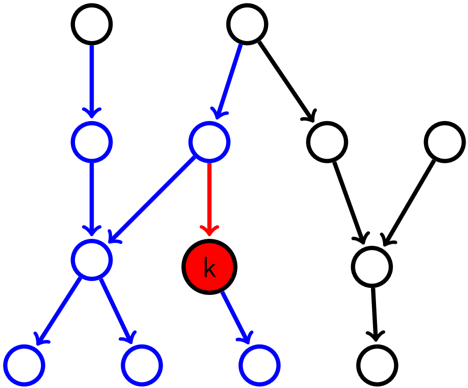


Sparseness Illustration - Total Output from node



$$u_k = (1 - \sigma_k)X_k - \sigma_k(\mathcal{X}_a - X_k)$$

Sparseness Illustration - Total input to node



$$v_k = (1 - \sigma_k/\alpha)X_k - \sigma_k/\alpha(\mathcal{X}_a - X_k)$$

Thank You!

Questions?

Price Lemmas

Lemma

For a node k with children set c_k and parent set p_k we have at any stationary point

$$\lambda_{j_1,k}[t] = \lambda_{j_2,k}[t] \quad \forall j_1, j_2 \in c_k, \ \forall t$$
$$\lambda_{k,i_1}[t] = \lambda_{k,i_2}[t] \quad \forall i_1, i_2 \in p_k, \ \forall t$$

The price on each outgoing link is the same.

Lemma

For any $j \in c_k$ and $i \in p_k$ we have that, at gradient zero,
$\lambda_{j,k}[t+1] = \lambda_{k,i}[t]$ *for all* $t \geq 0$, $t+1 \leq T-1$.

What a node buys at time t can later be sold at $t+1$

Price Lemmas

Lemma

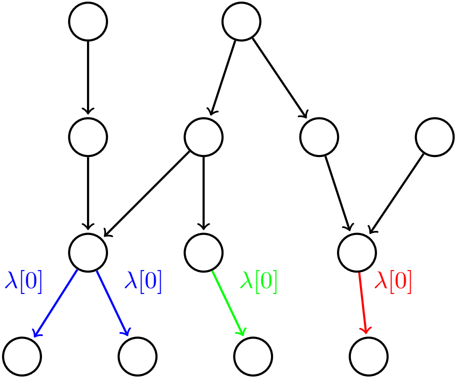
Consider a block $\mathcal{P}_a$ in layer l. For any outgoing links in that block we have that

$$\lambda_{k_1,j_1}[t] = \lambda_{k_2,j_2}[t] \quad \forall k_1, k_2 \in \mathcal{P}_a \ and \ j_1 \in c_{k_1}, j_2 \in c_{k_2}$$

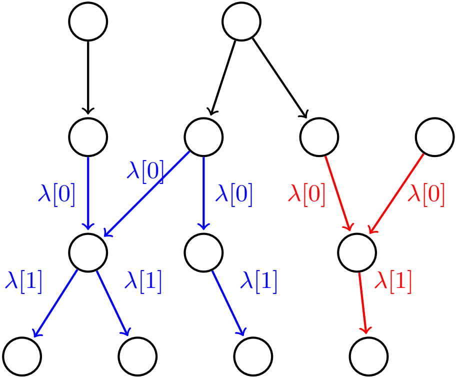
for all $t \leq (N-1) - (d-l)$

The outgoing price from each block is the same.

Lambda Illustration



lambda illustration



Lambda illustration