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Multiple Model Minimax Estimation

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Outline

1. Multiple Model
   Adaptive Estimation
2. Bounded operator norm
3. Multiple Model Minimax Filters
4. Future Research
5. Summary

Multiple Model Adaptive Estimation

- Estimate properties of a system using a finite set of feasible models.
- Model complex engineering systems as sets of linear systems
- State estimation
- Fault Detection

Example: F-16 — Maybeck

- Design with sensor and actuator redundancy
- One model for each sensor/actuator fault
- One model for healthy operation
- MMAE to detect sensor fault and compensate estimate
- MMAE to detect actuator fault and reroute control
- Fault detection and compensation < 1 second

1763	Bayesian Estimation
1795	Least Square Estimation
1906	Markov Process
1908	Fisher Information Theory
1929	Lattice Filter
1930	Monte Carlo method
1945	Cramer-Rao Bound
1949	Wiener Kolmogorov Filter
1960	Kalman Filter
1965	**Multiple Models Filter**
1981	$\mathcal{H}_\infty$ filter
1984	Interacting Multiple Model
1993	Particle Filter
1997	Unscented Kalman Filter
2000	Gauss-Hermite Filter
2003	Gaussian Particle Filter
2007	Smooth Variable Structure Filter
2009	Cubature Kalman Filter

Gausian noise — Magill, Lainiotis

Given matrices $F_1, \ldots, F_K$, $H_1, \ldots, H_K$, covariance matrices $Q_1, \ldots Q_K$ and $R_1, \ldots R_K$. Assume the dynamics can be described by

$$x_{t+1} = F_i x_t + w_t$$
$$y_t = H_i x_t + v_t$$
$$w_t \sim \mathcal{N}(0, Q_i), \quad v_t \sim \mathcal{N}(0, R_i).$$

The probability of i being active is

$$p(i|y_t, \ldots, y_0) = \frac{\overbrace{p(y_t|i, y_{t-1}, \ldots y_0)}^{\text{computable}} \overbrace{p(i|y_{t-1} \ldots y_0)}^{\text{recursion}}}{\sum_{j=1}^{K} \left(p(y_t|j, y_{t-1}, \ldots y_0) p(j|y_{t-1} \ldots y_0)\right)}.$$

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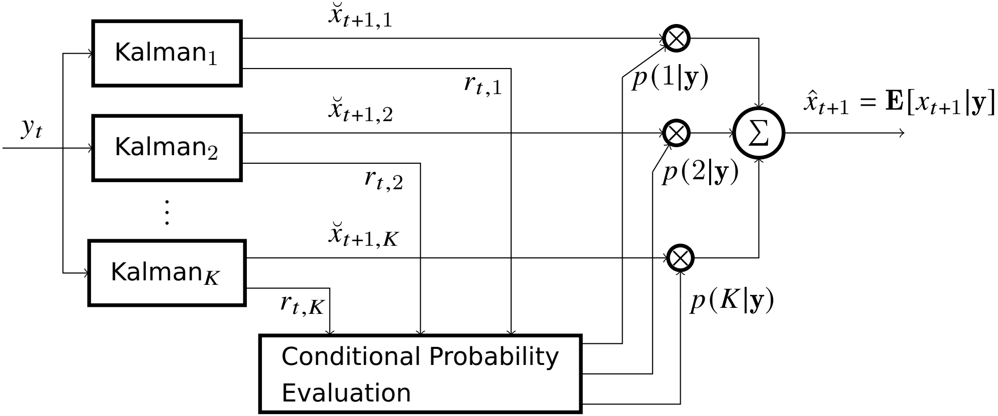
$$

Block Diagram



Jump-linear systems — Bar-Shalom

The discrete variable can jump!

$$
\begin{aligned}
x_{t+1} &= F(\delta_t)x_t + w_t \\
y_t &= H(\delta_t)x_t + v_t \\
p(\delta_{t+1}) &= Ap(\delta_t) \leftarrow \text{Markov Chain} \\
w_t \sim \mathcal{N}(0, Q_i), &\quad v_t \sim \mathcal{N}(0, R_i)
\end{aligned}
$$

- Enumerating each feasible trajectory $\neq$ tractable
- Interacting Multiple Model algorithm most used
- Mode probabilities $\longrightarrow$ Filtering update $\longrightarrow$ mixing probabilities $\longrightarrow$ overall prediction

Weaknesses

- **Weakness:** Strong assumptions on gaussian white noise
- **Solution:** Use Interacting multiple models with particle filter
- **New weakness:** Complex and computationally heavy

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Hardy space $\mathcal{H}^\infty$

$\mathcal{H}^\infty$ is a closed space with functions that are analytical and bounded inside the unit disc (open right half-plane). Normed space

$$\|G\|_\infty := \sup_{|z|<1} \bar{\sigma}[G(z)],$$

and

$$\|Gf\|_2 \leq \|G\|_\infty \|f\|_2.$$

Let $G : (w, v) \mapsto (\hat{x} - x)$ — map from disturbances to estimation error.

Idea: Find G with smallest $\|G\|_\infty$.

Godfrey Harold Hardy

1877 – 1944

- British mathematician
- number theory and analysis
- 1940 essay "A Mathematician’s Apology"

“It is never worth a first-class person’s time to express a majority opinion. By definition, there are plenty of others to do that.” — *paraphrased*

- Fredric Riesz
- 1880 – 1956
- Coined "Hardy Space"
- Teaching with style

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- Harald Cramér
- 1893 - 1985
- Ph. D — M. Riesz
- Cramér-Rao lower bound...

$\mathcal{H}^\infty$-filter

- Find filter with smallest ℓ^2 (L^2) gain from disturbance to error.
- Introduced by Mike Grimble 1987 — Fundamentals by Zames 1981.

$$\min_{\hat{x}} \max_{w,v,x_0} \frac{\sum |\hat{x}_t - x_t|^2}{|\hat{x}_0 - x_0|^2_{P_0^{-1}} + \sum(|w_t|^2_{Q^{-1}} + |v_t|^2_{R^{-1}})}$$

- Different characterizations — different results

Linear-Quadratic Game — Shen, Deng

- Algorithm for solving

$$\min_{\hat{x}} \max_{w,v,x_0} \left\{ \sum |\hat{x}_t - x_t|^2 - \gamma^2 \left(\hat{x}_0 - x_0|^2_{P_0^{-1}} + \sum (|w_t|^2_{Q^{-1}} + |v_t|^2_{R^{-1}}) \right) \right\}$$

- Guarantees $\|G\|_\infty < \gamma$. Bisect over γ.
- Multiple Model Adaptive Estimation?

Minimax criterion

$$x_{t+1} = F_i x_t + w_t$$
$$y_t = H_i x_t + v_t$$
$$\min_{\hat{x}_N} \max_{w,v,x_0,i} \left\{ |\hat{x}_N - x_N|^2 - \gamma^2 \left(|\hat{x}_0 - x_0|^2_{P_0^{-1}} + \sum_{t=0}^{N-1} \left(|w_t|^2_{Q^{-1}} + |v_t|^2_{R^{-1}} \right) \right) \right\}$$

Forward dynamic programming

- $w_t = x_{t+1} - F_i x_t$
- $v_t = y_t - H_i x_t$
- $\mathbf{y}^t = \{y_{t-1}, \ldots, y_0\}$

$$\min_{w,v,x_0} |\hat{x}_0 - x_0|^2_{P_0^{-1}} + \sum_{t=0}^{N-1} (|w_t|^2_{Q^{-1}} + |v_t|^2_{R^{-1}})$$

$$= \min_{x_{N-1},\ldots,x_0} |\hat{x}_0 - x_0|^2_{P_0^{-1}} + \sum_{t=0}^{N-1} (|x_{t+1} - F_i x_t|^2_{Q^{-1}} + |y_t - H_i x_t|^2_{R^{-1}})$$

$$=: V_N(x_N, \mathbf{y}^N)$$

Solution 1

- Solve using forward dynamic programming, i.e.

$$V_{t+1}(x_{t+1}, \mathbf{y}^{t+1}) = \min_{x_t} \left\{ |x_{t+1} - F_i x_t|^2_{Q^{-1}} + |y_t - H_i x_t|^2_{R^{-1}} + V_t(x_t, \mathbf{y}^t) \right\}$$

$$V_t(x_t, \mathbf{y}^t) = |\breve{x}_{t,i} - x_t|^2_{P_{t,i}^{-1}} + \underbrace{\sum_{k=0}^{t-1} |y_t - H_i \breve{x}_{t,i}|^2_{(R+H_i P_{t,i} H_i^\top)^{-1}}}_{c_{t,i}}$$

- $\breve{x}_{t,i}$ is a Kalman Filter prediction for the ith model
- $P_{t,i}$ is the error covariance of a kalman filter for the ith model
- We have compressed the measurement history.

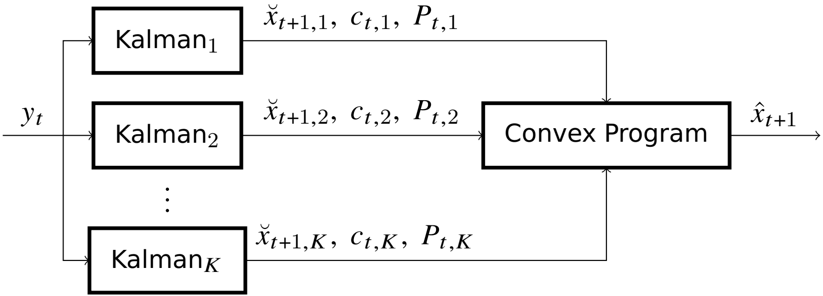
Solution 2

$$\min_{\hat{x}_N} \max_{w,v,x_0,i} \left\{ |\hat{x}_N - x_N|^2 - \gamma^2 \left(|\hat{x}_0 - x_0|^2_{P_0^{-1}} + \sum_{t=0}^{N-1} \left(|w_t|^2_{Q^{-1}} + |v_t|^2_{R^{-1}} \right) \right) \right\}$$

$$= \min_{\hat{x}_N} \max_{x_N, i} \left\{ |\hat{x}_N - x_N|^2 - \gamma^2 |\breve{x}_{N,i} - x_N|^2_{P_{t,i}^{-1}} - c_{N,i} \right\}$$

$$= \min_{\hat{x}_N} \max_{i} |\hat{x}_N - \breve{x}_{N,i}|^2_{(I - \gamma^{-2} P_{N,i})^{-1}} - \gamma^2 c_{N,i}$$

- Convex program
- Solve using CVX, or JuMP with Mosek or Hypatia.jl


$\breve{x}_{t+1,1},\ c_{t,1},\ P_{t,1}$
Kalman$_1$
y_t
$\breve{x}_{t+1,2},\ c_{t,2},\ P_{t,2}$
Kalman$_2$
Convex Program
$\hat{x}_{t+1}$
$\breve{x}_{t+1,K},\ c_{t,K},\ P_{t,K}$
Kalman$_K$

Future Research

- Multiple Model Adaptive Estimation with a bound on ℓ^2 gain.
- Uncountable parameter space.
- Output feedback minimax adaptive control. $\leftarrow$ connect with Anders' research.

Summary

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Resources

- Minimax Adaptive Estimation for Finite Sets of Linear Systems (Olle and Anders, 2021): https://arxiv.org/abs/2103.02479
- Julia package for recreating results and playing around: https://github.com/kjellqvist/MinimaxEstimation.jl
- Great review article — Gaussian Filters for Parameter and State Estimation (Afshari, et al., 2017): A General Review of Theory and Recent Trends: https://www.sciencedirect.com/science/article/pii/S0165168417300014
- Standard reference for all things related to linear filtering: Optimal Filtering by Anderson & Moore
- Maybecks case study (1999): https://doi.org/10.1002/(SICI)1099-1239(19991215)9:14<1051::AID-RNC452>3.0.CO;2-0