On H-infinity Structured Static State Feedback

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Outline

1. H-infinity Structured Static State Feedback
2. Comparison between non-structured and structured
3. Proposition

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H-infinity Static State Feedback

Let a LTI plant G be given in state-space by

$$\dot{x} = Ax + Eu + Bv$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$ and disturbance $v \in \mathbb{R}^q$.

H-infinity Static State Feedback

Find a static state feedback controller L such that the closed-loop system with state-space realization

$$\dot{x} = (A + EL)x + Bv$$

i is stable

ii and $||G_{cl,v\to z}(L)||_\infty < \gamma$

where z is performance output with cost matrix Q_1 on states x and Q_2 on input u, i.e., $x^T Q_1 x + u^T Q_2 u$.

$\rightarrow$ Bounded real lemma

Bounded real lemma

The closed-loop system

i is stable

ii and $||G_{cl,v\to z}(L)||_\infty < \gamma$

if and only if there exist a matrix L and a symmetric matrix $P > 0$ such that

$$\begin{bmatrix} (A+EL)^T P + P(A+EL) & PB & I & L^T \\ B^T P & -\gamma^2 I & 0 & 0 \\ I & 0 & -Q_1^{-1} & 0 \\ L & 0 & 0 & -Q_2^{-1} \end{bmatrix} < 0$$

Bounded real lemma

Trick: Right- and left-multiply by $\mathrm{diag}(P^{-1}, I, I)$.

$$\begin{bmatrix} WA^T + AW + EZ + Z^T E & B & W & Z^T \\ B^T & -\gamma^2 I & 0 & 0 \\ W & 0 & -Q_1^{-1} & 0 \\ Z & 0 & 0 & -Q_2^{-1} \end{bmatrix} < 0$$

where $W = P^{-1}$ is symmetric positive definite and $Z = LP^{-1}$.
L is given by $L = ZW^{-1}$.

H-infinity Structured Static State Feedback

[Tanaka, Langbort 2011].
Given plant G and assume that B is entry-wise non-negative. Then there exists a static state feedback controller

$$L \in \mathscr{L} = \{L \in \mathbb{R}^{m \times n} : L^j \in \mathcal{E}_j \;\; for \;\; all \;\; j = 1, \ldots, n\}$$

such that the closed-loop system

- i *is stable*
- ii *internally positive*
- iii *and* $||G_{cl,v \to z}(L)||_\infty < \gamma$

if and only if there exists a diagonal matrix $W > 0$ and a matrix $Z \in \mathscr{L}$ such that the LMI is feasible and $AW + EZ$ is Metzler.

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Specific type of systems

Consider a LTI system with states $x \in \mathbb{R}^n$, control inputs $u \in \mathbb{R}^m$, disturbance signals $v \in \mathbb{R}^n$ and state-space realization

$$\dot{x} = -\mathrm{diag}(a)x + Eu + Bv$$

where $a \in \mathbb{R}^n_{>0}$, $E \in \mathbb{Z}^{n \times m}$ and $B \in \mathbb{R}^{n \times n}_{\geq 0}$.

Moreover, each column of E has one entry equal to 1 and one entry equal to -1, while the remaining ones are zero.

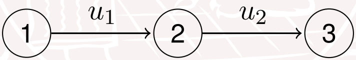
Problem formulation

Compare non-structured static state feedback with structured.

Network description

$$\dot{x} = -\operatorname{diag}(a_1, a_2, a_3)x + \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} u + Bv$$

where $a_i > 0$ and with associated graph



The arrow-head on the link depicts the positive direction of u_i.
However, the quantity goes in both direction, just with opposite sign.

Structured Static State Feedback

$$L = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \end{pmatrix}$$

Decentralized: $l_{13} = 0$ and $l_{21} = 0$.

$\mathbf{A} + \mathbf{EL}$ **Metzler:** $l_{11} \geq 0$, $l_{12} \leq 0$, $l_{22} \geq 0$ and $l_{23} \leq 0$.

Because $W > 0$ diagonal and $L = ZW^{-1}$, the constraints on L become linear constraints on Z.

Comparison of non-structured and structured

- Same bound γ
- In some cases, dependent on A and B, the structured static state feedback can be made even more sparse

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Proposition

The two convex problems,

$$\begin{aligned}&\text{variables } W \text{ symmetric, } Z\\&\text{minimize } \gamma^2\\&\text{subject to } W > 0,\ LMI(\gamma^2, W, Z) < 0\end{aligned}$$

and

$$\begin{aligned}&\text{variables } W \text{ diagonal, } Z\\&\text{minimize } \gamma^2\\&\text{subject to } W > 0,\ LMI(\gamma^2, W, Z) < 0, Z \in \mathcal{L}_{DP}\end{aligned}$$

give the same optimal value for this type of systems.

Conclusion

- Proposition: same bound on norm, proof?
- Performance needs to be compared more carefully
- Other types of systems where this still holds
- Searching for suitable "real" systems