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Fundamental limitations on the control of lossless systems

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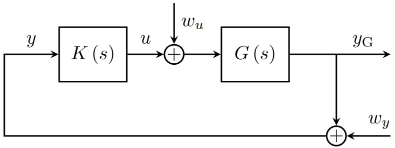
Lossless systems

- Used for describing transportation of physical commodities
- For example Power systems and other transportation systems
- $G(s) + G(-s)^{\mathsf{T}} = 0$, i.e. the Nyquist curve is only on the imaginary axis.
- The poles and zeros are only on the imaginary axis and interchanging.

State-space description of a lossless system



$$\dot{x} = Ax + B(u + w_u),\ x(0) = 0,$$
$$z = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \qquad (1)$$
$$y = Cx + D(u + w_u) + w_y,$$

and

$$\dot{x}_K = A_K x_K + B_K y,\ x_K(0) = 0,$$
$$u = C_K x_K + D_K y, \qquad (2)$$

Denote the closed-loop transfer function from w to z as defined by equations (1) and (2) as $T_{zw}(s)$. Find

$$\gamma_\bullet^* = \inf\{\gamma : ||T_{zw}(s)||_\bullet < \gamma\},$$

where $||\cdot||_\bullet$ denotes either the H_2 or H_∞ norm.

Main theorem

Assume that A, B, C, and D are defined as in the previous slide, and that the pair (A, B) is controllable. If there exists a positive definite P such that

$$PA + A^\mathsf{T} P = 0,\ PB = C^\mathsf{T}, \text{ and } D + D^\mathsf{T} = 0,$$

then in the H_2 case, $\gamma^*_{H_2} = \sqrt{2\mathrm{tr}(CB)}$,

and in the H_∞ case, $\gamma^*_{H_\infty} = \sqrt{2}$.

- Modifying the problem to allow for nonlinear control laws will not reduce $\gamma^*_{H_2}$ and $\gamma^*_{H_\infty}$. The theorem gives a fundamental limit on achievable performance over all causal control laws, not just finite dimensional linear-time-invariant control laws in the form of equation (2)

Implications for power systems

In the absence of damping, after linearisation, the swing equation power system model is described by the differential-algebraic equations

$$M_k\ddot{\theta}_k = p_{\mathrm{N},k} + u_k + w_{u,k}, \begin{bmatrix}\theta_k(0)\\ \dot{\theta}_k(0)\end{bmatrix} = 0, k \in \{1,\dots,n\},$$

$$\begin{bmatrix}p_{\mathrm{N}}\\ 0\end{bmatrix} = -\begin{bmatrix}K_{\mathrm{a}} & K_{\mathrm{b}}\\ K_{\mathrm{b}}^{\mathsf{T}} & K_{\mathrm{c}}\end{bmatrix}\begin{bmatrix}\theta\\ \theta_{\mathrm{int}}\end{bmatrix}.$$

The Schur complement of K gives the Kron reduced lapalcian:

$K_{\mathrm{red}} = K_{\mathrm{a}} - K_{\mathrm{b}}K_{\mathrm{c}}^{-1}K_{\mathrm{b}}^{\mathsf{T}}$, which can be factored as $K_{\mathrm{red}} = LL^{\mathsf{T}}$, where $L \in \mathbb{R}^{n\times n-1}$

$$\begin{aligned}\dot{x} &= \begin{bmatrix}0 & -M^{-1}L\\ L^{\mathsf{T}} & 0\end{bmatrix}x + \begin{bmatrix}M^{-1}\\ 0\end{bmatrix}(u + w_u),\ x(0) = 0,\\ y &= \begin{bmatrix}I & 0\end{bmatrix}x + w_y.\end{aligned} \tag{3}$$

The main theorem with the power system model

$$\gamma^*_{H_2} = \sqrt{2\text{tr}(CB)} = \sqrt{2\left(\frac{1}{M_1} + \cdots + \frac{1}{M_n}\right)}$$

$$\gamma^*_{H_\infty} = \sqrt{2}$$

- $\gamma^*_{H_2}$ is highly affected by the inertia constants.
- $\gamma^*_{H_\infty}$ doesn't depend on any process parameter
- With more renewables like solar and wind power introduced
  - The total inertia will decrease.
  - Some nodes will have very little inertia.
  - More stochastic disturbances are introduced

Case study with the optimal controllers

Form the proof of $\gamma^*_{H_\bullet}$ the optimal controllers fall out. These are:

- For the H_2-norm: $K(s) = -C(sI - A + 2BC)^{-1}B$
- For the H_∞-norm: $K(s) = -\sqrt{2}I$

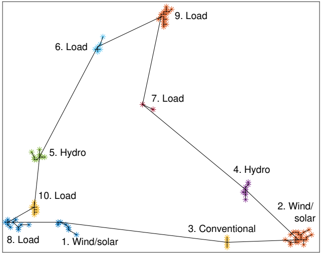
Defining the gains form local disturbance $\tilde{w}_k$ to local output $\tilde{z}_k$ as

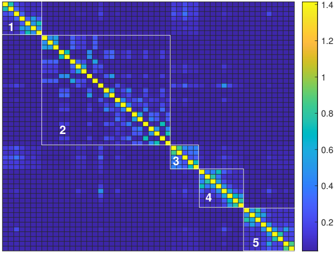
$$\gamma_{H_2,ik} = ||T_{\tilde{z}_i\tilde{w}_k}(s)||_{H_2} \qquad \gamma_{H_\infty,ik} = ||T_{\tilde{z}_i\tilde{w}_k}(s)||_{H_\infty}$$

where

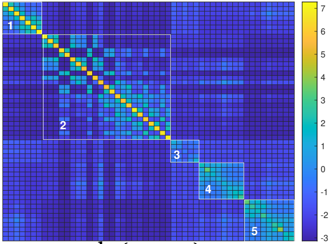
$$\tilde{w}_k = \begin{bmatrix} w_{u,k} \\ w_{y,k} \end{bmatrix} \text{ and } \tilde{z}_k = \begin{bmatrix} Cx + Du_k \\ u_k \end{bmatrix}$$

Case study and results

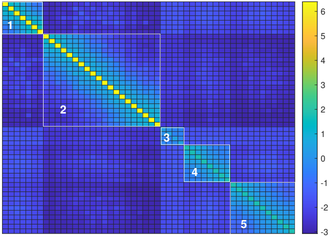




$\gamma_{H_\infty,ik}$

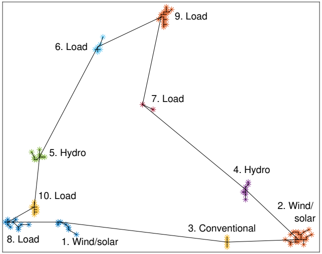


$\ln(\gamma_{H_2,ik})$



$\ln(\bar{\gamma}_{H_2,ik})$

Lumped models



	1	2	3	4	5
1	5.8	-1	-1	-0.8	-0.2
2	-1	4.6	-1	-0.2	-0.2
3	-1	-1	-1	-0.9	-0.9
4	-0.8	-0.2	-0.9	0.6	0.5
5	-0.2	-0.2	-0.9	0.5	1.2

$\ln(\gamma_{H_2,ik})$

	1	2	3	4	5
1	1.4	0.4	0.4	0.4	0.4
2	0.4	1.4	0.4	0.6	0.4
3	0.4	0.4	1.4	0.4	0.4
4	0.4	0.6	0.4	1.4	0.8
5	0.4	0.4	0.4	0.8	1.4

$\gamma_{H_\infty,ik}$

- In power systems it is common to work with aggregated models.
- In our case this corresponds to lumping the clusters to 1 equivalent node.

- $\gamma_{H_2,ik}$ is reduced while $\gamma_{H_\infty,ik}$ is not.

Conclusions

- For a state space representation of a lossless system the H_2 norm of a disturbance affect on the output is given by $\gamma^*_{H_2} = \sqrt{2\mathrm{tr}(CB)}$, and for the H_∞ norm it is $\gamma^*_{H_\infty} = \sqrt{2}$.
- In a lossless power system model, the effects are highly local both in H_2 and H_∞.
- The largest effects of disturbances are seen in nodes with small inertia, such as renewables like solar and wind, but these effects do not spread in any major sense to other nodes or clusters.

Questions?