(Almost) Global UAV Control System Design Aided by Lyapunov, Barbalat, and Matrosov

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5/21-2021

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Difficulties

- Nonlinear
- Nonautonomous
- Configuration manifolds

Main idea

- Example 1 Peaking
- Example 2 Uniform stability
- Example 3 Case study



Figure 1: The small Crazyflie quadrotor

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Pontus: Good idea to talk about Lyapunov functions!

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[1] E. Lefeber, M. Greiff, and A. Robertsson, "Filtered output feedback tracking control of a quadrotor UAV," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 5764–5770, 2020

Overview and main thread of presentation

- Introduction
 - Pitfalls in NLTV analysis
 - Introducing the case study
- Lyapunov's Second Method
 - The main idea
- Barbălat's Lemma
 - The main idea
 - Useful variations
 - Application
- Matrosov's Theorems
 - The main idea
 - Application
- Simulation example



Figure 2: Example simulation (to be explained)

Introduction - A note of caution

Some warnings

- Quite dense
- Lots of signals...!
- Some omitted details
- Σ A system (with memory)
- ${m R}$ A rotation (always $\in \mathbb{R}^{3 imes 3}$)
- ·r A reference
- ·e A tracking error
- G Global, as in globally stable (GS)
- A Asymptotic, as in asymptotically stable (AS)
- E Exponential, as in *exponentially* stable (ES)
- U Uniform, as in *uniformly* stable (US)



Consider two linear systems

$$\Sigma'_1: \quad \dot{\boldsymbol{x}}_1 = \boldsymbol{A}_1 \boldsymbol{x}_1, \qquad \qquad \boldsymbol{x}_1(t_\circ) = \boldsymbol{x}_{1\circ} \tag{1a}$$

$$\Sigma_2: \quad \dot{\boldsymbol{x}}_2 = \boldsymbol{A}_2 \boldsymbol{x}_2, \qquad \qquad \boldsymbol{x}_2(t_\circ) = \boldsymbol{x}_{2\circ}. \tag{1b}$$



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Consider the cascade

$$\Sigma_1: \quad \dot{x}_1 \triangleq A_1 x_1 + B x_2.$$
 (2)

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 (1a)
 $\Sigma_2: \quad \dot{x}_2 = A_2 x_2, \qquad \qquad x_2(t_\circ) = x_{2\circ}, \qquad \qquad (1b)$

Consider the cascade

$$\Sigma_1: \quad \dot{\boldsymbol{x}}_1 \triangleq \boldsymbol{A}_1 \boldsymbol{x}_1 + \boldsymbol{B} \boldsymbol{x}_2. \tag{2}$$

If $\{\Sigma_1', \Sigma_2\}$ are asymptotically stable (AS), then $\{\Sigma_1, \Sigma_2\}$ is AS, as

$$\{\Sigma_1, \Sigma_2\}: \quad \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{A}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix}$$
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If $\{\Sigma_1', \Sigma_2\}$ are asymptotically stable (AS), then $\{\Sigma_1, \Sigma_2\}$ is AS, as

$$\{\Sigma_1, \Sigma_2\}: -\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{A}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix}$$
(3)

What if the systems are nonlinear and non-autonomous?

Consider two nonlinear systems

$$\begin{split} \Sigma_1' : & \dot{x}_1 = f_1(t, x_1), & x_1(t_\circ) = x_{1\circ} & (4a) \\ \Sigma_2 : & \dot{x}_2 = f_2(t, x_2), & x_2(t_\circ) = x_{2\circ}. & (4b) \end{split}$$

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Consider the cascade

$$\Sigma_1: \quad \dot{\boldsymbol{x}}_1 \triangleq f_1(t, \boldsymbol{x}_1) + g(t, \boldsymbol{x}_1, \boldsymbol{x}_2) \boldsymbol{x}_2.$$

(5)

Consider two nonlinear systems

$$\Sigma'_1: \quad \dot{x}_1 = f_1(t, x_1), \qquad \qquad x_1(t_\circ) = x_{1\circ}$$
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Consider the cascade

$$\Sigma_1: \quad \dot{\boldsymbol{x}}_1 \triangleq f_1(t, \boldsymbol{x}_1) + g(t, \boldsymbol{x}_1, \boldsymbol{x}_2) \boldsymbol{x}_2. \tag{5}$$

If $\{\Sigma'_1, \Sigma_2\}$ are asymptotically stable (AS), then $\{\Sigma_1, \Sigma_2\}$ is...

... it depends!

Example (Peaking)

Let $t_{\circ} = 0$, and consider a nonlinear system

$$\begin{split} \Sigma_1' &: \dot{x}_1 = -x_1^3, \\ \Sigma_2 &: \dot{x}_2 = -x_2, \\ \end{split} \qquad \begin{array}{l} x_1(t_\circ) = x_{1\circ}, \\ x_2(t_\circ) = x_{2\circ}. \\ \end{array} \end{split}$$

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$$\Sigma_1 : \dot{x}_1 = -(1 - x_2)x_1^3$$

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The solution for the system $\{\Sigma_1, \Sigma_2\}$ is

$$x_1(t) = \operatorname{sign}(x_{1\circ})(x_{1\circ}^{-2} + 2x_{2\circ}(e^{-t} - 1) + 2t)^{-1/2}$$

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Here Σ'_1 is GAS, and Σ_2 is GAS, but for their cascade through g, the solution $x_1(t)$ diverges with a finite escape time even for $x_{1\circ} > 0$.



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If $\{\Sigma_1', \Sigma_2\}$ are asymptotically stable (AS), then $\{\Sigma_1, \Sigma_2\}$ is...

GAS if a set of sufficient conditions on $\{\Sigma'_1, \Sigma_2\}$ and g are met.

Example (Peaking)

Here Σ'_1 is GAS, and Σ_2 is GAS, but for their cascade through g, the solution $x_1(t)$ diverges with a finite escape time even for $x_{1\circ} > 0$.



Growth rate: If $\{\Sigma'_1, \Sigma_2\}$ is AS, and there exists continuous $\theta_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $||g(x_1, x_2)|| \leq \theta_1(||x_2||) + \theta_2(||x_2||)||x_1||$, then $\{\Sigma_1, \Sigma_2\}$ is also AS (see e.g. Panteley '99 [2], or Loria '05 [3]).

Takeaway

In the nonlinear setting, the separation principle generally does not apply. Care must be taken when connecting a found controller with an observer, as the introduced dynamics may cause the states to diverge, even if $\{\Sigma'_1, \Sigma_2\}$ has very nice properties. Especially true when aiming for global or almost global stability properties.



Takeaway

In the nonlinear setting, the separation principle generally does not apply. Care must be taken when connecting a found controller with an observer, as the introduced dynamics may cause the states to diverge, even if $\{\Sigma'_1, \Sigma_2\}$ has very nice properties. Especially true when aiming for global or almost global stability properties.

With a "nice" feedback, a "nice" estimator and "good" connection, are asymptotic stability properties enough? What about robustness?

Consider two non-autonomous systems:

$$\begin{split} \Sigma : \quad \dot{\boldsymbol{x}} &= f(t, \boldsymbol{x}), \\ \Sigma_{\Delta} : \quad \dot{\boldsymbol{x}} &= f(t, \boldsymbol{x}) + \boldsymbol{\Delta}(t, \boldsymbol{x}), \end{split}$$

where $\|\boldsymbol{\Delta}(t, \boldsymbol{x})\| \leq L$ for all $t \geq t_{\circ}$. What does Σ say about on Σ_{Δ} ?



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Example (Loria, Panteley, Teel '99)

Consider a nominal system with defined by $a(t) = (t+1)^{-1}$, with

$$\dot{x} = f(t, x) = \begin{cases} -a(t) \operatorname{sign}(x) & \text{if } |x| > a(t) \\ -x & \text{if } |x| \le a(t) \end{cases}, \quad t \ge t_{\circ} \ge 0.$$

When adding a $\Delta(t) = L \neq 0$, solutions grow unbounded as $t \to \infty$. In fact, $\lim_{t\to\infty} x(t)/t = \pm L$ (depending on the sign of $x(t_\circ)$) Problem: the solution of x(t) for the unperturbed system $\dot{x} = f(t, x)$ depends on $(t_{\circ}, x(t_{\circ}))$, the convergence to the origin is not *uniform*.



Problem: the solution of x(t) for the unperturbed system $\dot{x} = f(t, x)$ depends on $(t_{\circ}, x(t_{\circ}))$, the convergence to the origin is not *uniform*. Solution: Require *uniform* asymptotic stability (independent of t_{\circ}).



Problem: the solution of x(t) for the unperturbed system $\dot{x} = f(t, x)$ depends on $(t_{\circ}, x(t_{\circ}))$, the convergence to the origin is not *uniform*.

Solution: Require *uniform* asymptotic stability (independent of t_{\circ}).

Takeaway

In general, to say something about the robustness properties of systems on the form Σ : $\dot{x} = f(t, x)$ we require uniform stability properties. Several local or global boundedness results follow (see e.g., Khalil '96 [4, Theorem 3.18 combined with Lemma 4.3]).

The attitude dynamics of the UAV

$$\Sigma : \begin{cases} \dot{R} = RS(\omega), \\ J\dot{\omega} = S(J\omega)\omega + \tau, \end{cases}$$
(Controlled system) (6a)

$$\Sigma_r : \begin{cases} \dot{R}_r = R_r S(\omega_r), \\ J\dot{\omega}_r = S(J\omega_r)\omega_r + \tau_r, \end{cases}$$
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$$J \in \mathbb{R}^{3 \times 3}$$
 s.t. $J = J^{\top} \succ 0$

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$$\begin{aligned} \boldsymbol{J} \in \mathbb{R}^{3 \times 3} & \text{s.t.} \quad \boldsymbol{J} = \boldsymbol{J}^\top \succ \boldsymbol{0} \\ \boldsymbol{S} : \mathbb{R}^3 \to \mathbb{R}^{3 \times 3} & \text{s.t.} \quad \boldsymbol{S}(\boldsymbol{a}) \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b} \end{aligned}$$

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Objective: Find $g({m R}, \omega, {m R}_r, \omega_r, {m au}_r)$ such that ${m R} o {m R}_r, \omega o \omega_r$



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Many (almost) global solutions exist [5]–[8], however...

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- Essentially a full-state feedback requires an estimator

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- Many (almost) global solutions exist [5]–[8], however...
- Essentially a full-state feedback requires an estimator
- Stability should to be uniform, estimator needs to be (almost) globally stabilizing, interconnection needs to satisfy conditions.

Alternatively, solve a *filtered output feedback problem*, as in [1].



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• Define a filter memory $\boldsymbol{\zeta}$ (here $\{\hat{\boldsymbol{R}}, \hat{\boldsymbol{\omega}}\} \in \mathrm{SO}(3) imes \mathbb{R}^3$).

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- Define an update of $\boldsymbol{\zeta}$ in a set of measurements $\{\boldsymbol{y}_i\}_{i=1}^N$ (IMU).

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- Define a filter memory $\boldsymbol{\zeta}$ (here $\{\hat{\boldsymbol{R}}, \hat{\boldsymbol{\omega}}\} \in \mathrm{SO}(3) imes \mathbb{R}^3$).
- Define an update of $\boldsymbol{\zeta}$ in a set of measurements $\{\boldsymbol{y}_i\}_{i=1}^N$ (IMU).
- Define a feedback law $g(\zeta, \mathbf{R}_r, \omega_r, \tau_r)$ such that $\mathbf{R} \to \mathbf{R}_r, \omega \to \omega_r$ and $\{\zeta, \mathbf{R}, \omega, \tau\}$ remain bounded.

Case: filtered output feedback in [1].

- Illustrate the ideas
- Less focus on precision
- Defining the errors
- A curious Lyapunov function
- Applying Barbălat
- Applying Matrosov
- Simulation example

Available solites at your sal			
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Filtered Output Fee	Iback Tracking Control		
of a Quadrotor UAV			
Erjen Leicher ' Marvas Greiff '' Anders Esheriman ''			
Technology, PO Rev 101, 340 (r-mail: 4.4.	 Emderson, The Netherlands (Lepher One of) 		
 Department of Automatic Con (s-mail /Marcus Creff, Au 	irei, Land University, Land, Steelen Irea Referitance/Docated.26, ar)		
Abstract: We prevent a tracking controlle	for qualitator UAVs which uses partial state		
information and filters the measurements to measurable and band measurable stability of	attenuate units. We show uniform almost global the condition cheesed here ensines, which involve		
releasing against househol disturbances. We	illustrair the preformance of the controller by		
means of several monetical reamples, include	g a complex looping maneurer.		
Copyright C 2021 The Andrew, Thirth on open an	rea article sailer the CC III ACAD lisener		
Kroneth UWs tracking output ImBack (nind, andarar showers. Loanar methods		
In this many we consider the scaling of construction a	local stability reads for their controller. In Abdemanner		
catedrolector calination for the tracking control	and Tayris (2000) and Zon (2006) the non-zero view		
of qualities U.O.s. without the use of linear velocity	and differential system when successfully he this score a		
would like to attranate theor using a like blowers	saturate only the combined proportional and differenti-		
and since for acollarur systems the restainty equivalence principle does not held, controller-observer combinations need to be carefully collesigned.	 mated action. Furthermore, in these two papers stability producer limitand using Earbidia's Lemma, showing out asymptotic stability, not uniform asymptotic stability. 		
Starting with the work of Carownile and Villani (1998) origin feedback lows that usite the tracking problem is such the utilities downing how four dama of the	we do in this paper. Only the latter guarantees releasing against bounded perturbations, ef. Panteley et al. (1998 and (Khalil, 2000, Lemma 9.3). Also, in Zen (2016) time		
Asl and Yoon (2015) an origin feedback for only the	deviation of the vistal costed axis are used in th attitude materials, histolucing the area for measuring		
the inner loop for the attitude dynamics is last enough	tion). In Mulmanural and Tarrhi (2020 the desire a		
incorrect, an manually proof for the resuming overall sphere	the attitude materilles has here done in quaternism. J		
aught to represent the attitude, resulting in singularities	both the quaternions q and -q represent the same att both the resultion stitleds controlles now arbitit the s		
and to tar to cause "gamma lock", making their approach fail for complex trajectories with large angular more acuity	called dynamical survivaling behavior, are Blast and Ber- siste (2000). Nuclei of of the doors controllers are sto		
such as the looping maneuror considered in this paper.	measurements directly in the controller, i.e., unliveral.		
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er an (perce), an ear new association of the authors only these (second of) without consider on extent facilities	to provat an output indicate for the tracking mate		
tracking problem: Abdemaneral and Tayota (2020). Zor	business in demanding rough paragraphic		
(2016), and Shao et al. (2018). These papers, as well as	 only illowed signals are used in the control action (ill memory and action is thereby discussion). 		
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specifying the desired attitude, a non-zero vietual contex	· proportional and derivative actions of the transle		
action is required for the vistand controller. In Shareri al (2008) this a not guaranteed by the proposed controller	nately (which is browlinked if they have opposite signs		
The meanth leading to those results has territed basing from the	Furthermore, we consider the attitude on NO(2) instea of mine Folds could (which from simulation in some		
Sandak Science Frankluss (SFF) propert "Smanler support of visual antipotion for smart robust" (BETL/1000) and the ELLET functions Control on Local Distances.	in many source approximation and insight builts and in analogue		

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Case study - Errors

Consider the errors,

$oldsymbol{R}_e = oldsymbol{R}_r oldsymbol{R}^ op$	$G \in SO(3),$	(7a)
$ ilde{m{R}} = \hat{m{R}} m{R}^ op$	$M \odot \in \mathrm{SO}(3),$	(7b)
$\omega_e=\omega_r-\omega$	$\in \mathbb{R}^3,$	(7c)
$ ilde{\omega} = \hat{\omega} - \omega$	$\in \mathbb{R}^3,$	(7d)
$\hat{oldsymbol{\omega}}_e = oldsymbol{\omega}_r - \hat{oldsymbol{\omega}}$	$\in \mathbb{R}^3$.	(7e)

Case study - Errors

Consider the errors,

$oldsymbol{R}_e = oldsymbol{R}_r oldsymbol{R}^ op$	\in SO(3),	(7a)
$ ilde{m{R}} = \hat{m{R}} m{R}^ op$	\in SO(3),	(7b)
$\omega_e = \omega_r - \omega$	$\in \mathbb{R}^3,$	(7c)
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$\hat{\omega}_e = \omega_r - \hat{\omega}$	$\in \mathbb{R}^3$.	(7e)

With the controller and observer in [1] (here omitted for brevity),

$$\dot{oldsymbol{R}}_e = f_1(t, oldsymbol{R}_e, ilde{oldsymbol{R}}_e, oldsymbol{\omega}_e, ilde{oldsymbol{\omega}})$$
 (8a)

$$ilde{m{R}}_e = f_2(t, m{R}_e, ilde{m{R}}_e, m{\omega}_e, ilde{m{\omega}})$$
 (8b)

$$\boldsymbol{J}\dot{\boldsymbol{\omega}}_{e} = f_{3}(t, \boldsymbol{R}_{e}, \tilde{\boldsymbol{R}}_{e}, \boldsymbol{\omega}_{e}, \tilde{\boldsymbol{\omega}})$$
 (8c)

$$oldsymbol{J}\dot{ extbf{\omega}}=f_4(t,oldsymbol{R}_e, ilde{oldsymbol{R}}_e,oldsymbol{\omega}_e, ilde{oldsymbol{\omega}})$$
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$\tilde{\omega} = \hat{\omega} - \omega$	$\in \mathbb{R}^3,$	(7d)
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$$\boldsymbol{J}\dot{\boldsymbol{\omega}} = f_4(t, \boldsymbol{R}_e, \tilde{\boldsymbol{R}}_e, \boldsymbol{\omega}_e, \tilde{\boldsymbol{\omega}})$$
 (8d)

Consider an AS linear system,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x}(t_{\circ}) \in \mathbb{R}^{n}$$

Then, there exists a solution

$$oldsymbol{A}^{ op}oldsymbol{P}+oldsymbol{P}oldsymbol{A}+oldsymbol{Q}=oldsymbol{0}, \quad oldsymbol{P}=oldsymbol{P}^{ op}\succoldsymbol{0}, \quad oldsymbol{Q}=oldsymbol{Q}^{ op}\succoldsymbol{0},$$

using lyap(A,Q) and a quadratic Lyapunov function $\mathcal{V} = x^{\top} P x$.



Consider a PD function in the errors (where $k_i > 0$, $v_i \not\parallel v_j \in \mathbb{R}^3$),

$$\mathcal{V}_1 = \sum_{i=1}^N \frac{k_i}{2} \| R_e \tilde{R}^\top v_i - v_i \|^2 + \frac{1}{2} \omega_e^\top J \omega_e + \sum_{i=1}^N \frac{k_i}{2} \| \tilde{R} v_i - v_i \|^2 + \frac{1}{2} \tilde{\omega}^\top J \tilde{\omega}.$$

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The function is NSD along the solutions of the error dynamics, with

$$\dot{\mathcal{V}}_1 = -c_R \bigg\| \sum_{i=1}^N k_i S(\hat{\boldsymbol{R}}^\top \boldsymbol{v}_i) (\boldsymbol{R}_r^\top \boldsymbol{v}_i + \boldsymbol{R}^\top \boldsymbol{v}_i) \bigg\|^2 - \boldsymbol{\omega}_e^\top \boldsymbol{K}_{\boldsymbol{\omega}} \boldsymbol{\omega}_e - \tilde{\boldsymbol{\omega}}^\top \boldsymbol{C}_{\boldsymbol{\omega}} \tilde{\boldsymbol{\omega}}.$$

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(i) Standard Lyapunov theory is difficult to apply

(ii) As $\dot{\mathcal{V}}_1$ is negative semi-definite, \mathcal{V}_1 is upper bounded,

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- (iv) Due to (iii), $\dot{\mathcal{V}}_1$ is uniformly continuous in time

Lemma (Barbălat '59 [9])

Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a uniformly continuous function on its domain. If $\Phi(t) = \lim_{t \to \infty} \int_0^t \phi(\tau) d\tau$ exists and is finite, $\phi(t) \to 0$ as $t \to \infty$.



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Application of the Lemma yields asymptotic convergence to

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Lemma (Variant of Barbălat's Lemma [10, Lemma 2.2.12])

Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be any differentiable function. If f(t) converges to zero as $t \to \infty$ and its derivative satisfies

$$\dot{f}(t) = f_0(t) + \eta(t) \qquad t \ge 0,$$
(9)

where $f_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is uniformly continuous and $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}$. If $\eta(t)$ tends to zero as $t \to \infty$, $\dot{f}(t)$ and $f_0(t)$ tend to zero as $t \to \infty$.



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$$oldsymbol{J}\dot{oldsymbol{\omega}}_e=f_3(t,oldsymbol{R}_e,oldsymbol{\omega}_e, ilde{oldsymbol{\omega}})$$

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Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be any differentiable function. If f(t) converges to zero as $t \to \infty$ and its derivative satisfies

$$\dot{f}(t) = f_0(t) + \eta(t) \qquad t \ge 0,$$
(9)

where $f_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is uniformly continuous and $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}$. If $\eta(t)$ tends to zero as $t \to \infty$, $\dot{f}(t)$ and $f_0(t)$ tend to zero as $t \to \infty$.

Consider

 $oldsymbol{J} \dot{oldsymbol{\omega}}_e = oldsymbol{S}(oldsymbol{J}oldsymbol{\omega}) oldsymbol{\omega}_r - oldsymbol{K}_{oldsymbol{\omega}}(oldsymbol{\omega}_e - ilde{oldsymbol{\omega}}) - \sum_{i=1}^N k_i oldsymbol{S}(oldsymbol{R}_r^{ op} oldsymbol{v}_i) \hat{oldsymbol{R}}^{ op} oldsymbol{v}_i,$

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$$\underbrace{\boldsymbol{J}\dot{\boldsymbol{\omega}}_{e}}_{\triangleq \dot{\boldsymbol{f}}(t)} \underbrace{\boldsymbol{S}(\boldsymbol{J}\boldsymbol{\omega})\boldsymbol{\omega}_{e} + \boldsymbol{S}(\boldsymbol{J}\tilde{\boldsymbol{\omega}})\boldsymbol{\omega}_{r} - \boldsymbol{K}_{\boldsymbol{\omega}}(\boldsymbol{\omega}_{e} - \tilde{\boldsymbol{\omega}})}_{\triangleq \eta(t)} - \underbrace{\sum_{i=1}^{N} k_{i}\boldsymbol{S}(\boldsymbol{R}_{r}^{\top}\boldsymbol{v}_{i})\hat{\boldsymbol{R}}^{\top}\boldsymbol{v}_{i}}_{\triangleq -f_{0}(t)}.$$

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Consider

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As $f(t) \to 0$ and $f_0(t)$ is uniformly continuous, $f_0(t) \to 0$ as $t \to \infty$.

Summary from Barbălat

• First application,

$$\sum_{i=1}^{N} k_i \boldsymbol{S}(\hat{\boldsymbol{R}}^{\top} \boldsymbol{v}_i) (\boldsymbol{R}_r^{\top} \boldsymbol{v}_i + \boldsymbol{R}^{\top} \boldsymbol{v}_i) \triangleq f_0(t) + g_0(t) \to \boldsymbol{0}.$$

• Signal chasing, $f_0(t)
ightarrow 0 \Rightarrow g_0(t)
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Case study - Barbălat

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All solutions converge to an invariant set

$$\mathcal{S} = \left\{ (\boldsymbol{R}_{e}, \tilde{\boldsymbol{R}}, \boldsymbol{\omega}_{e}, \tilde{\boldsymbol{\omega}}) \in \mathrm{SO}(3)^{2} \times \mathbb{R}^{6} \left| \begin{array}{c} \sum_{i=1}^{N} k_{i} \boldsymbol{S}(\boldsymbol{R}_{r}^{\top} \boldsymbol{v}_{i}) \hat{\boldsymbol{R}}^{\top} \boldsymbol{v}_{i} = \boldsymbol{0} \\ \sum_{i=1}^{N} k_{i} \boldsymbol{S}(\boldsymbol{R}^{\top} \boldsymbol{v}_{i}) \hat{\boldsymbol{R}}^{\top} \boldsymbol{v}_{i} = \boldsymbol{0} \\ \boldsymbol{\omega}_{e} = \boldsymbol{0} \\ \tilde{\boldsymbol{\omega}} = \boldsymbol{0} \end{array} \right\} \right\}$$

Case study - Barbălat

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$$\sum_{i=1}^{N} k_i \boldsymbol{S}(\hat{\boldsymbol{R}}^{\top} \boldsymbol{v}_i) (\boldsymbol{R}_r^{\top} \boldsymbol{v}_i + \boldsymbol{R}^{\top} \boldsymbol{v}_i) \triangleq f_0(t) + g_0(t) \to \boldsymbol{0}.$$

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However, convergence to ${\cal S}$ is asymptotic, but not necessarily *uniform*.

The main idea of Matrosov

- $\bullet~\text{NSD}~\dot{\mathcal{V}}_1$ and non-autonomous error dynamics
- Find uniformly bounded function \mathcal{Y}_i which upper bounds \mathcal{V}_i
- Satisfy nested properties on \mathcal{Y}_i
- Exact details in [3, Thm. 1 and Thm. 2].



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In this context, let

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• $\mathcal{V}_2 = \boldsymbol{\omega}_e^\top \sum_{i=1}^N k_i \boldsymbol{S}(\boldsymbol{R}_r^\top \boldsymbol{v}_i) \hat{\boldsymbol{R}}^\top \boldsymbol{v}_i.$

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- Then, plugging in the error dynamics,

$$\frac{\mathrm{d}\mathcal{V}_2}{\mathrm{d}t} \leq - \left\| \sum_{i=1}^N k_i \boldsymbol{S}(\boldsymbol{R}_r^{\top} \boldsymbol{v}_i) \hat{\boldsymbol{R}}^{\top} \boldsymbol{v}_i \right\|^2 + M_3 \left\| \begin{bmatrix} \boldsymbol{\omega}_e \\ \tilde{\boldsymbol{\omega}} \end{bmatrix} \right\| + M_4 \left\| \begin{bmatrix} \boldsymbol{\omega}_e \\ \tilde{\boldsymbol{\omega}} \end{bmatrix} \right\|^2 \triangleq \mathcal{Y}_2,$$

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•
$$\mathcal{Y}_1=0\Rightarrow \boldsymbol{\omega}_e=\tilde{\boldsymbol{\omega}}=\mathbf{0}\Rightarrow \mathcal{Y}_2\leq 0,$$
 and

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$$\dot{\mathcal{V}}_1 \triangleq \mathcal{Y}_1$$

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• $\mathcal{Y}_1 = 0 \Rightarrow \boldsymbol{\omega}_e = \tilde{\boldsymbol{\omega}} = \mathbf{0} \Rightarrow \mathcal{Y}_2 \leq 0$, and • $\mathcal{Y}_1 = \mathcal{Y}_2 = 0 \Rightarrow (\boldsymbol{R}_e, \tilde{\boldsymbol{R}}, \boldsymbol{\omega}_e, \tilde{\boldsymbol{\omega}}) \rightarrow \mathcal{S}.$

The main idea of Matrosov

- $\bullet~\text{NSD}~\dot{\mathcal{V}}_1$ and non-autonomous error dynamics
- Find uniformly bounded function \mathcal{Y}_i which upper bounds \mathcal{V}_i
- Satisfy nested properties on \mathcal{Y}_i
- Exact details in [3, Thm. 1 and Thm. 2].



Case study - Summary

Barbalat + Signal chasing + Matrosov

• Uniform asymptotic convergence of $(\mathbf{R}_e, \tilde{\mathbf{R}}, \omega_e, \tilde{\omega}) \rightarrow S$

It is also possible to show

- By conditions on $\{(k_i, v_i)\}_{i=1}^N$: S contains 13 isolated equilibrium points
- By local linearization
- $:(\boldsymbol{I},\boldsymbol{I},\boldsymbol{0},\boldsymbol{0})\in\mathcal{S}$ is UAGAS
- By local linearization $(I, I, 0, 0) \in S$ is ULES







Recall, the proposed Lyapunov function

$$\mathcal{V}_1 = \sum_{i=1}^N \frac{k_i}{2} \| \boldsymbol{R}_e \tilde{\boldsymbol{R}}^\top \boldsymbol{v}_i - \boldsymbol{v}_i \|^2 + \frac{1}{2} \boldsymbol{\omega}_e^\top \boldsymbol{J} \boldsymbol{\omega}_e + \sum_{i=1}^N \frac{k_i}{2} \| \tilde{\boldsymbol{R}} \boldsymbol{v}_i - \boldsymbol{v}_i \|^2 + \frac{1}{2} \tilde{\boldsymbol{\omega}}^\top \boldsymbol{J} \tilde{\boldsymbol{\omega}}.$$



• The Lyapunov function time derivative in the errors (black)

$$\dot{\mathcal{V}}_1 = -c_R \Big\| \sum_{i=1}^N k_i oldsymbol{S}(\hat{oldsymbol{R}}^{ op} oldsymbol{v}_i) (oldsymbol{R}_r^{ op} oldsymbol{v}_i + oldsymbol{R}^{ op} oldsymbol{v}_i) \Big\|^2 - \omega_e^{ op} oldsymbol{K}_{oldsymbol{\omega}} \omega_e - ilde{oldsymbol{\omega}}^{ op} oldsymbol{C}_{oldsymbol{\omega}} ilde{oldsymbol{\omega}}.$$

• And evaluated from \mathcal{V}_1 by numerical differentiation (blue)



Conclusions

Summary

- Separation principle and peaking
- Uniform stability and robustness
- Tools from Lyapunov, Barbalat, and Matrosov
- Making sense of 27 error signals
- Code: AerialVehicleControl.jl [11]
- More: ACC Wed, 10.15 and 11.00 (UTC -5)

Thank you for listening!

[11] M. Greiff, AerialVehicleControl.jl, nonlinear and robust UAV control system synthesis, github.com/mgreiff/AerialVehicleControl.jl, 2020

References

- E. Lefeber, M. Greiff, and A. Robertsson, "Filtered output feedback tracking control of a quadrotor UAV," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 5764–5770, 2020.
- [2] E. Panteley and A. Loria, "On global uniform asymptotic stability of nonlinear time-varying systems in cascade," *Systems & Control Letters*, vol. 33, no. 2, pp. 131–138, 1998.
- [3] A. Loria, E. Panteley, D. Popovic, and A. R. Teel, "A nested matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems," *IEEE Transactions on automatic control*, vol. 50, no. 2, pp. 183–198, 2005.
- [4] H. Khalil, "Khalil, nonlinear systems," *Prentice-Hall, Inc., New Jersey*, 1996.

References II

- [5] T. Lee, M. Leok, and N. H. McClamroch, "Geometric tracking control of a quadrotor UAV on SE(3)," in 49th IEEE Conference on Decision and Control (CDC), 2010, pp. 5420–5425.
- [6] —, "Nonlinear robust tracking control of a quadrotor UAV on SE(3)," *Asian Journal of Control*, vol. 15, no. 2, pp. 391–408, 2013.
- T. Lee, "Global exponential attitude tracking controls on SO(3)," IEEE Transactions on Automatic Control, vol. 60, no. 10, pp. 2837–2842, 2015.
- [8] M. Greiff, Z. Sun, and A. Robertsson, "Attitude control on SU(2): Stability, robustness, and similarities," *IEEE Control Systems Letters*, 2021. DOI: 10.1109/LCSYS.2021.3049440.
- I. Barbalat, "Systemes d'équations différentielles d'oscillations non linéaires," *Rev. Math. Pures Appl*, vol. 4, no. 2, pp. 267–270, 1959.

References III

- [10] A. A. J. Lefeber, *Tracking control of nonlinear mechanical systems*. Universiteit Twente Eindhove, The Netherlands, 2000.
- [11] M. Greiff, AerialVehicleControl.jl, nonlinear and robust UAV control system synthesis, github.com/mgreiff/AerialVehicleControl.jl, 2020.