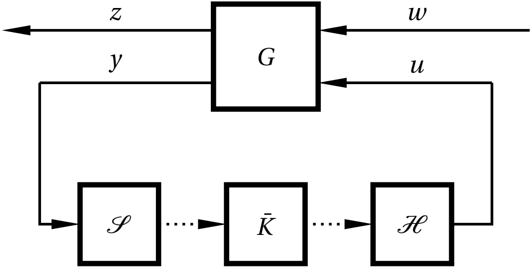
On Optimal Sampling for Event-Based Control

Marcus T. Andrén

Problem

How to optimally co-design controller structure and sampling strategy?



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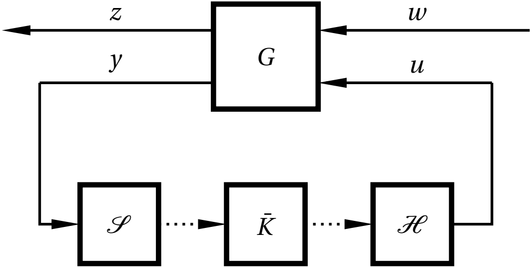

Goal

Minimize

$$\|T_{zw}\|_2^2 := \lim_{T\to\infty} E[\frac{1}{T}\int_0^T \|z(t)\|_2^2 dt]$$

w.r.t $\mathscr{S}, \bar{K}, \mathscr{H}$ and sampling

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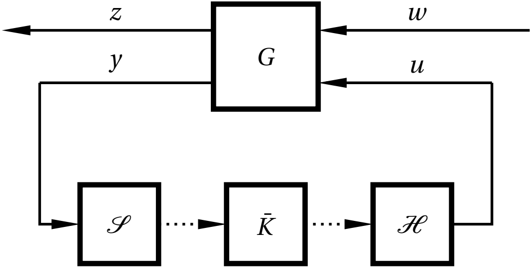

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Why?

- Most efficient use of sampling
- To lower energy consumption
- To reduce risk for congestion

Lower Bound on $\|T_{zu}\|_2^2$

$$G: \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right] \begin{bmatrix} x \\ w \\ u \end{bmatrix}$$

Without restrictions on sampling, analog LQG is optimal:

LQR Gain:

$$L = (D_{zu}^T D_{zu})^{-1}(B_u^T X + D_{zu}^T C_z)$$

Kalman Gain:

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with optimal cost

$$\|T_{zw}\|_2^2 = \gamma_0^2 = \mathrm{Tr}(B_w^T X B_w) + \mathrm{Tr}(C_z Y C_z^T) + \mathrm{Tr}(XAY + YA^T X)$$

- No sampled-data controller can beat γ_0^2
- How close can we get?

The Optimal Controller Structure

[Mirkin 2017, Th 5.1]

The optimal attainable $\mathscr{H}_2$ performance using sampled-data controllers for any sampling sequence $\{t_i\}$ is

$$\|T_{zw}\|_2^2 = \gamma_0^2 + \|H\|_2^2$$

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where H is a reset system driven by white process ϵ with intensity $D_{yw}D_{yw}^T$:

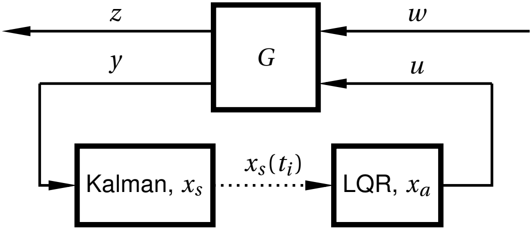
$$H: \begin{cases} \dot{x}_{\mathrm{H}}(t) = Ax_{\mathrm{H}}(t) - K\epsilon, & x_{\mathrm{H}}(t_i) = 0 \\ \eta(t) = -(D_{zu}^T D_{zu})^{\frac{1}{2}} L x_{\mathrm{H}}(t) \end{cases}$$

Degradation from sampling given by average variance of η!

The Optimal Sampled-Data Controller

[Mirkin 2017]

The optimal sampled-data controller can be realized as:



Sensor-side

$\dot{x}_s(t) = Ax_s(t) + B_u u(t) + K(y(t) - C_y x_s(t))$

Actuator-side

$\dot{x}_a(t) = (A - B_u L)x_a(t)$

$x_a(t_i) = x_s(t_i)$

$u(t) = -Lx_a(t)$

Remark: $x_{\mathrm{H}} = x_a - x_s$ and $\epsilon = y - C_y x_s$

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Structure in [Mirkin 2017] optimal for *any uniformly bounded* $\{t_i\}$

$\implies$ Sampling scheme can be considered separately!

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Define $\phi(x_{\text{H}}) : x_{\text{H}} \to \{0,1\}$ and let $\{t_i\} = \{t | \phi(x_{\text{H}}(t)) = 1\}$.

Optimal Sampling Goal:

$$\min_{\phi} \quad \underbrace{\|H\|_2^2}_{\text{Performance Degradation}} + \underbrace{\lim_{T\to\infty} \frac{1}{T} E[\sum_i \phi(x_{\text{H}}(t_i))]}_{\text{Average Sampling Rate}} \cdot \underbrace{\rho}_{\text{Per-sample-cost}}$$

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where ϵ is a white process with intensity R.

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Note: Identical to our optimization problem if:

$$Q = L^T D_{zu}^T D_{zu} L \qquad R = K D_{yw} D_{yw}^T K^T$$

Solution?

Optimal sampling problem can be re-formulated as finding a *value function* $V(x)$:

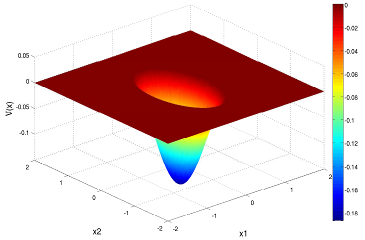
[Henningsson 2012, Paper II, Th 1]

Suppose a bounded function $V(x)$ and constant J are found satisfying

$$x^T Q x + x^T A^T \nabla V + \frac{1}{2}\mathrm{Tr}(R\nabla^2 V) \geq J \quad (1)$$

$$\rho \geq V(x) - V(0) \quad (2)$$

where for each x equality is achieved in either (1) or (2). Then the optimal cost is J and it's optimal to trigger when equality is achieved in (2).



Integrator Case: A = 0

[Henningsson 2012]

A closed-form solution exists for the integrator case:

$$V(x) = \begin{cases} -\frac{1}{4}g(x)^2, & g(x) \geq 0 \\ 0 & \text{else} \end{cases}$$

where

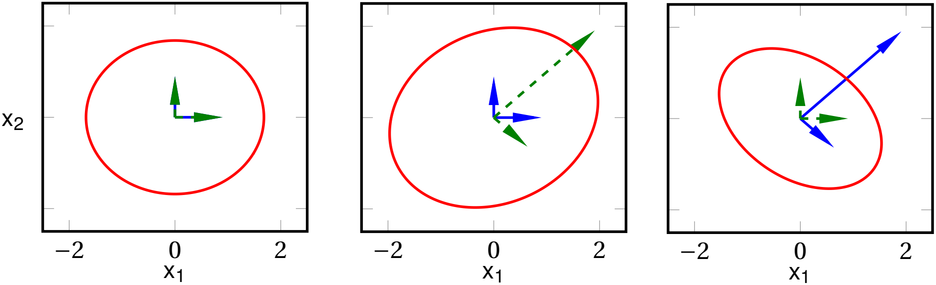
$$g(x) = 2\sqrt{\rho} - x^T P x$$

and P is the solution to the Riccati-like equation

$$PRP + \frac{1}{2}\text{Tr}(RP)P = Q$$

Integrator Case: A = 0

The optimal trigger bound is the ellipsoid $x^T P x = 2\sqrt{\rho}$.
"Shape" of P depends on Q and R (here, $\rho = 1$):



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Non-optimal vs. Optimal Sampling?

Let's look at a 2nd order integrator example:

$$\left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right] = \left[\begin{array}{c|cc} 0 & \begin{bmatrix}(R_d^{1/2}N_{\pi/8})^T & 0\end{bmatrix} & I \\ \hline \begin{bmatrix} N_{\pi/4}Q_d^{1/2} \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ I & \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{array}\right]$$

$$N_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \qquad R_d = Q_d = \text{diag}([1,\ 5])$$

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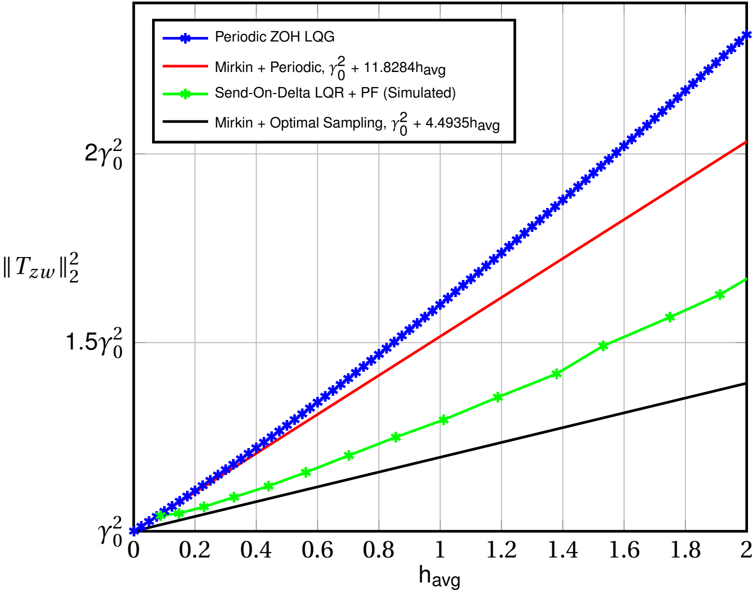
- Periodic ZOH LQG
- [Mirkin 2017] + Periodic Sampling
- LQR with Particle Filter + Send-On-Delta Sampling
- [Mirkin 2017] + Optimal Sampling

Example, A=0



Periodic ZOH LQG
Mirkin + Periodic, γ_0^2 + 11.8284h_{avg}
Send-On-Delta LQR + PF (Simulated)
Mirkin + Optimal Sampling, γ_0^2 + 4.4935h_{avg}
$2\gamma_0^2$
$1.5\gamma_0^2$
γ_0^2
$\|T_{zw}\|_2^2$
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
h_{avg}

Example, A=0

Using optimal controller structure, optimal sampling was $\frac{11.8284}{4.4935} \approx 2.63$ times more efficient than periodic.

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In fact, for any n^{th} order integrator system, the ratio is:

$$1 + \frac{2}{n} \leq \frac{\text{Periodic}}{\text{Optimal}} \leq 3$$

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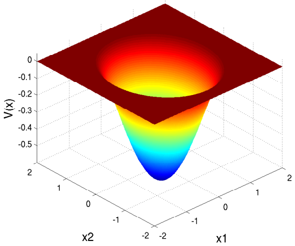
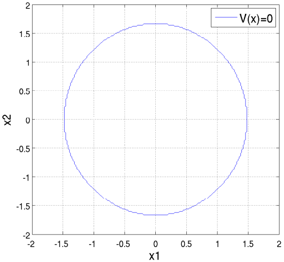
- The million \$ question (and current focus). Solving this $\Longrightarrow$ A synthesis method for event-based control!
- Currently we can obtain $V(x)$ for 2^{nd} order systems numerically.
- Any hypothesis based on numerical results can be checked using PDE for $V(x)$.

Numerically Obtained $V(x)$





$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

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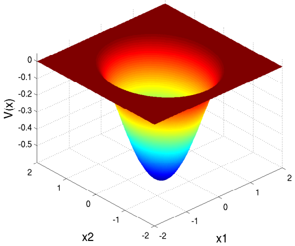

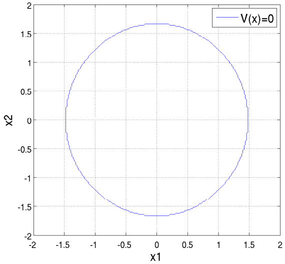

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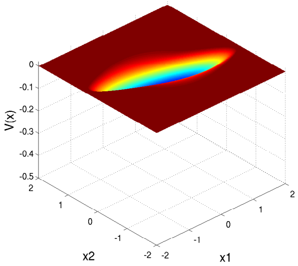
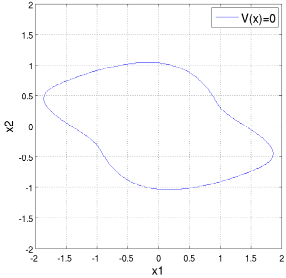
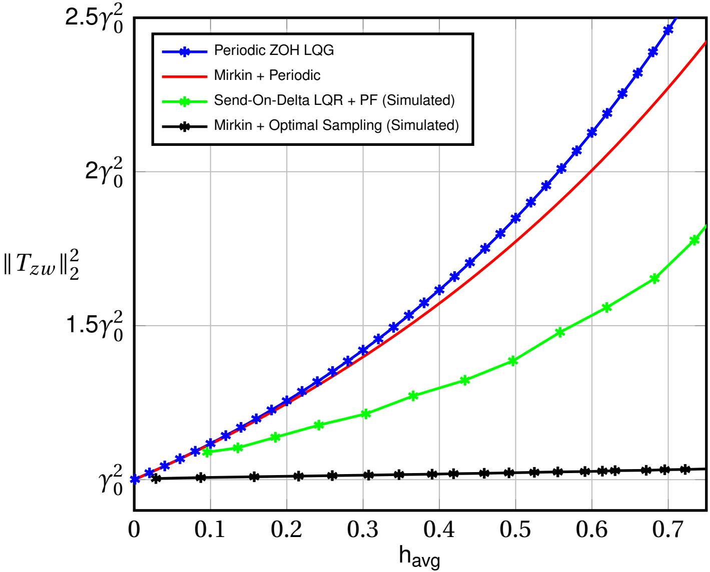




$$A = \begin{bmatrix} 1 & -1 \\ 0 & -70 \end{bmatrix}$$

Example, $A = \begin{bmatrix} 1 & -1 \\ 0 & -70 \end{bmatrix}$



Conclusion

- Problem: Co-design of controller and sampling strategy
- Nice result from Mirkin makes the problem separable.
- Similar work by Toivo directly applicable to the sampling problem.
- $A = 0$ case solved, $A \neq 0$ only numerically.
- Any hypothesis can be verified/rejected with PDE for $V(x)$.

References:

L. Mirkin,“Intermittent redesign of analog controllers via the Youla parameter,” IEEE Trans. Automat. Control, vol.62, 2017, (to appear). [Online]. Available: http://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=7523893

T. Henningsson, “Stochastic event-based control and estimation,” Ph.D. dissertation, Dept. of Automatic Control, Lund University, Lund, Sweden, 2012.